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A course in mathematics appreciation

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A COURSE IN MATHEMATICS

APPRECIATION

by
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A Thesis

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ABSTRACT

Daniel J. Kraft, A Course in Mathematics Appreciation, 1995, J. Sooy, Mathematics Education.

The purpose of this study is to create a mini-course in mathematics appreciation at the senior high school level.

The mathematics appreciation course would be offered as an elective to students in the 11th or 12th grade, who are concurrently enrolled in trigonometry or calculus.

The topics covered in the mathematics appreciation course include: systems of numeration, congruences, Diophantine equations, Fibonacci sequences, the golden section, imaginary numbers, the exponential function, pi, perfect numbers, numbers with shape, ciphers, magic squares, and root extraction techniques.

In this study, the student is exposed to mathematical proofs, where appropriate, and is encouraged to create practice problems for other members of the class to solve. Also, areas for research are suggested so that the student may explore, even more deeply, areas which hold a particular interest for that student.

These topics are treated with a three-pronged approach--historical, recreational, and practical. It is the author's contention, supported by research, that this approach, along with the choice of topics, will assist in developing and enhancing the mathematics potential of the student to the highest possible extent.

MINI-ABSTRACT

Daniel J. Kraft, A Course in Mathematics Appreciation, 1995, J. Sooy, Mathematics Education.

The purpose of this study is to create a mini-course in mathematics appreciation for 11th or 12th grade students who are concurrently enrolled in trigonometry or calculus.

The topics in this course are treated with a three-pronged approach--historical, recreational, and practical--which, according to research, will assist in developing and enhancing the mathematics potential of the student.

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CHAPTER 1

Introduction to Thesis

Introduction

In a typical high school course of study, there exists a core curriculum for each subject matter area. In the field of English, students typically follow a sequence consisting of English I, II, and III; in the field of science, students are offered biology, chemistry, and physics; and in the field of foreign language, students take three years of a selected language. In addition to the core curriculum, many students are offered the opportunity to select supplemental courses in various disciplines that enhance the student's understanding of the subject matter and offer an aesthetic approach that does not exist in the mainstream curriculum. For example, courses in English literature, astronomy, or French civilization afford the interested student an opportunity to study topics in a selected area for the sheer enjoyment of the subject. Unfortunately, a high school student is rarely afforded the opportunity for aesthetic pursuit in the field of mathematics.

Background

In the field of mathematics, the courses which normally constitute the high school core curriculum are algebra I and II, geometry, trigonometry, and calculus. In these courses, designed to prepare college-bound students for careers in mathematics, science, and other related fields, topics are encountered that develop skills required for successful mastery of higher mathematics. Rarely in the standard high school curriculum do students encounter courses that concentrate solely on the historical development, mathematical derivation, or aesthetic treatment of these topics.

Statement of Problem

The purpose of this study is to create a mini-course in mathematics appreciation at the senior high school level.

Significance of the Problem

Because of the time constraints imposed by the scope and sequence of the aforementioned core courses in mathematics, the student is not afforded the opportunity to study ancillary topics such as pi, the golden ratio, magic squares or Pythagorean triples in great detail. A course in mathematics appreciation would enable the student to pursue many of the historical and cultural aspects of mathematics associated with these topics. While the core courses in mathematics provide the student with what could be called "vertical development", a course in mathematics appreciation would provide the student with the opportunity for "horizontal development." An example of the importance assigned to the philosophy behind such a course can be found in the description of the mathematics curriculum at Holy Family College in Philadelphia, which states: "To help instill an appreciation of the natural origin and evolutionary growth of the basic mathematical ideas."[1]

And in the book, Number: The Language of Science, Tobias Dantzig supports this philosophy by stating that " . . . our school curricula, by stripping mathematics of its cultural content and leaving a bare skeleton of technicalities, have repelled many a fine mind."[2]

Therefore, the purpose of this study is to create a mathematics elective intended for those students who wish to gain an appreciation of concepts not thoroughly developed in the typical mathematics core curriculum.

Limitations of the Study

The mathematics appreciation course should be offered as an elective to students in the 11th and 12th grade concurrently enrolled in trigonometry or calculus. The mathematics appreciation course is not intended to replace the mathematics core courses, but rather to enhance topics which are given only an ephemeral treatment in these courses.

Procedures

The topics that will be addressed in the mathematics appreciation course will include: numeration systems, congruence, Diophantine equations, Fibonacci sequences, the golden section, imaginary numbers, the exponential function, pi, perfect numbers, the shapes of numbers, cryptology, mystic arrays, and root extraction.These topics appear frequently in books on number theory and seem to be held in high regard by the authors. For example, in his book, The Mathematical Traveler, Calvin C. Clawson states, "Then we will free our imaginations and discover the strange transcendental numbers, such as π-- numbers so peculiar that we cannot even write them down."[3]

Each of these topics in the mini-course will be treated with a three-pronged approach. First, an historical background of each topic will be offered in order to enable the reader to grasp the deeper significance of the subject. The purpose of the historical approach is, as Kenneth H. Rosen states in his book dealing with number theory, "to emphasize that number theory has an old and rich history as well as a modern vitality."[4] Secondly, mathematical derivations or suggested activities dealing with the topic will be presented, many with a recreational approach. This approach is supported by Martin Gardner, who has written a book designed to stimulate popular interest in mathematics. In his book, Mathematical Puzzles and Diversions, Gardner states that " . . . popular interest in recreational mathematics has continued to increase."[5] Third, one or more practical applications of the topic will be suggested so that the student may discover its relevancies.

Notes

[1] Holy Family College, Catalog, 1994-96, 82.

[2] Tobias Dantzig, Number: The Language of Science (New York: The MacMillan Company, 1954), viii.

[3] Calvin C. Clawson, The Mathematical Traveler (New York: Plenum Press, 1994), 3.

[4] Kenneth H. Rosen, Elementary Number Theory (New York: Addison-Wesley Publishing Company, 1988), vi.

[5] Martin Gardner, Mathematical Puzzles and Diversions (New York: Simon and Schuster, 1961), 9.

CHAPTER 2

Review of Related Literature

Introduction

Research shows that a three-pronged approach (historical, recreational, and practical) to the teaching of mathematics has a great deal of merit. The various components of this approach have been supported by: authors of books dealing with assorted topics of mathematics; the Commission on Standards for School Mathematics; members of the educational community; and contributors to the annual yearbook publications of the National Council of Teachers of Mathematics (NCTM). It is also the author's contention that the utilization of this approach in a mathematics appreciation course will assist in developing and enhancing the mathematics potential of the student to the highest possible extent.

Review of the Literature

The review of the literature suggests that there is an interest in the relationship between understanding the history of certain mathematical concepts and the purposes that these concepts serve. The review also implies that a knowledge of these historical topics can further enhance the student's interest in the study of mathematics.

Mathematics does not exist in isolation. Historically, mathematics has influenced many fields and in turn has been influenced by many developments. It is research into these developments that contributes to the creation of a curriculum based on historical mathematical concepts. Myron Rosskopf offers that events in history, ranging from the settlement of the New World to the Industrial Revolution, have contributed to the development of curricula which sometimes emphasize education and at other times push it into the background.[1]

In 1969 the National Council of Teachers of Mathematics deemed the study of the history of mathematics important enough to create a yearbook on the use of the history of mathematics in the teaching of mathematics. The authors proposed that the curriculum should include topics of significant value for all grade levels and that it should encourage, "the teacher or the student to do further reading or study in the same or related topics."[2] They further expressed "hope that this will increase the interest of the students in mathematics and their appreciation for the cultural aspects of the subject."[3]

One of the dangers of teaching mathematics in isolation was expressed by Jacques Barzun (Teacher in America) who states, "I have more than an impression--it amounts to a certainty--that algebra is made repellent by the unwillingness or inability of teachers to explain why. . . . There is no sense of history behind the teaching, so the feeling is given that the whole system dropped down ready-made from the skies, to be used only by born jugglers."[4]

Another reason for a teacher to adopt the historical approach is that situations are created which afford the students the opportunity for discovering the relationship between the concrete and the abstract on their own. "A sufficiently concrete and detailed tracing of the history of the development of a generalized idea is one of the best ways to teach an appreciation of the nature and role of generalization and abstraction."[5]

The panel of the NCTM cautions that a proper balance between the historical approach and the modern curriculum should be maintained. "The important thing is neither to throw out all that is old nor to add whatever is new but to develop and pass on to our students new syntheses of old ideas and systems as well as to introduce new concepts and systems that are appropriate. Insight into the development (history) of ideas can serve to improve both the curriculum maker's choices and the teacher's power to communicate insights and stimulate interest."[6] Tom Kieren appears to agree when he observes that although current society learns from the past, it does not necessarily do new things with the same topics, but it certainly does things in a different way. The process has been an evolution which has taught people to value the past.[7]

In 1986, the Board of Directors of NCTM established the Commission on Standards for School Mathematics. The Standards is a document designed to establish a framework for school mathematics and to determine what the mathematics curriculum should include in terms of emphasis and priority. Through this document, the NCTM panel has encouraged educators "to focus attention on the need for student awareness of the interaction between mathematics and the historical situations from which it has developed and the impact that interaction has on our culture and our lives."[8] This panel also advocates mathematics as a concentration which can contribute to the better understanding of many other disciplines and content areas.[9] The contributors believe that this method of approaching mathematics will "enhance the students' self-concepts as well as their attitudes toward, and interest in, mathematics."[10]

The standards of the NCTM call for three key items to be examined in all grade levels of mathematics: "communicating, connecting, and valuing mathematics. History allows us to study all three."[11] It is a way to humanize the study of numbers. It allows students to see relevancy and therefore, perhaps, to become interested in the study of mathematics and to want to investigate further. The excitement created can only enrich the study of mathematics.[12]

As important as it is to consider the past when developing a regimen for the present, it is also necessary to contemplate the future. Robert Swain, in his article Modern Mathematics and School Arithmetic, exhorts the educator to take a serious look at what the future of mathematics may hold and to plan accordingly. He sees possible attention being given to such topics as Diophantine problems and modular arithmetic. He hopes that students will think of mathematics as a functional tool, as well as an end in itself.[13]

When one is considering the creation of a high school mathematics curriculum, it is necessary to keep in mind the values which the different areas of mathematics possess. These values may include, but not be limited to, the use of mathematics in the future, the merits of mathematics in other fields, and mathematics for its own sake. Cain, Carry, and Lamb suggest that each high school must select those goals, values, and priorities that are valid for itself. They believe that mathematics for its own sake would target approximately the top 10 percent of the student population. These are the students who would benefit the most from expansion into such an area of mathematics.[14]

The needs of these students must be addressed. The recreational approach could prove to be an effective way to meet these needs. "Teaching so that students understand the 'whys,' teaching for meaning and understanding, teaching so that children see and appreciate the nature, role, and fascination of mathematics, teaching so that students know that men are still creating mathematics and that they too may have the thrill of discovery and invention--these are objectives eternally challenging, ever elusive."[15]

In the core curriculum, the student is often faced with a great deal of tedium in an attempt to master the curriculum. In the proposed mini-course in mathematics appreciation, the student would be afforded the opportunity to experience the recreational approach. Jan Mokros writes in The Education Digest that students are not able to mature in mathematics sufficiently if they have a steady diet of only rote learning of numbers. They lack a varietal approach which would tend to stimulate them to go further in their mathematics study. They need to experience their own strategies and discoveries, both the successes and the failures. "They never experience the aesthetic high of inventing theorems or get to explain and defend the problem-solving strategies. Simply put, they neither do nor experience much mathematics."[16]

The study of the history of mathematics and the development of different number systems assist in illustrating the relationship between mathematics and the practical world. There are those who draw a connection between mathematics and the worlds of music, art, and science.[17] "If mathematics is an art, some appreciation of this fact, and of the relation of mathematics to the world of physical reality, can be as much a part of the liberal education of a doctor, lawyer, or average intelligent citizen as is some appreciation of the humanities."[18]

The relationship between mathematics and the real world should enable the student to appreciate the value of mathematics. In her review of Street Mathematics and School Mathematics, Wendy L. Millroy states that there has long been a fascination with the relationship between the mathematics learned in the schools and that which is used outside the confines of the school.[19]

Others have felt strongly about the relationship between mathematics and the real world. In his review of an 1811 mathematics textbook, Frank J. Swetz points out the emphasis on the practical applications of mathematics. The text stresses the importance in the society of the day of doing arithmetic and solving problems.[20]

The panel of the NCTM cautions that the connections between the study of mathematics and the applications to the practical world are not always obvious nor always expected. It is the teacher's task to assist the student in determining these connections.[21]

Therefore, curriculum development and revision can be a very difficult task. Koeno Gravemeijer suggests that "compiling a curriculum is comparable to solving a jigsaw puzzle, akin to taking pieces of history to form a coherent whole in the present, with possibly a new perspective."[22] Once the philosophy of the curriculum has been determined, the next task is to determine which topics should be included for study.

Notes

[1] Myron F. Rosskopf, "Mathematics Education: Historical Perspectives," The Teaching of Secondary School Mathematics, (The National Council of Teachers of Mathematics, 1970): 3-15.

[2] Historical Topics for the Mathematics Classroom, (The National Council of Teachers of Mathematics, 1969): ix.

[3] Historical Topics, x.

[4] Jacques Barzun, Teacher in America, cited in Historical Topics for the Mathematics Classroom, (The National Council of Teachers of Mathematics, 1969): 1.

[5] Phillip S. Jones, "The History of Mathematics as a Teaching Tool," Historical Topics for the Mathematics Classroom, (The National Council of Teachers of Mathematics, 1969): 13.

[6] Jones, 17.

[7] Carolyn Kieran, "Doing and Seeing Things Differently: A 25-Year Retrospective of Mathematics Education Research on Learning," Journal For Research in Mathematics Education, 25, no.6 (December 1994): 604.

[8] Curriculum and Evaluation Standards for School Mathematics, (The National Council of Teachers of Mathematics,1989): 6.

[9] Curriculum and Evaluation Standards, 11.

[10] Curriculum and Evaluation Standards, 131.

[11] James K. Bidwell, "Humanize Your Classroom with the History of Mathematics," Mathematics Teacher, (September 1993): 461.

[12] Bidwell, 464.

[13] Robert L. Swain, "Modern Mathematics and School Arithmetic," Instruction in Arithmetic (The National Council of Teachers of Mathematics, 1960): 292-294.

[14] Ralph W. Cain, L. Ray Carry, and Charles E. Lamb, "Mathematics in Secondary Schools: Four Points of View," The Secondary School Mathematics Curriculum (The National Council of Teachers of Mathematics, 1985): 22-23.

[15] Jones, 1.

[16] Jan Mokros, "Math Textbooks: Where's the Math?" The Education Digest, (Nov. 1994): 62.

[17] Jones, 9.

[18] Jones. 9-10.

[19] Wendy L. Millroy, "Exploring the Nature of Street Mathematics," Journal for Research in Mathematics Education, 25, no.3 (May 1994): 304.

[20] Frank J. Swetz, "Back to the Present: Ruminations on an Old Arithmetic Text," Mathematics Teacher, (Sept. 1993): 493-494.

[21] Jones, 11.

[22] Koena Gravemeijer, "Educational Development and Developmental Research in Mathematics Education," Journal for Research in Mathematics Education, 25, no.5 (November 1994): 447.

CHAPTER 3

Procedures

Introduction

The selection of the following topics for the mathematics appreciation course is a result of their inclusion in numerous books on number theory and other branches of mathematics. Additional testimony to the importance of these topics is provided by numerous articles written by the mathematical community and found in publications such as those written by the National Council of Teachers of Mathematics. The student enrolled in the mathematics appreciation course will be afforded the opportunity to view each of the selected topics through historical, recreational, and practical lenses.

Procedures

The following topics have been selected for inclusion in the mathematics appreciation course along with the rationale for their selection.

Numeration Systems

The first topic in the mathematics appreciation course will involve various systems of numeration. Working in different numeration systems involves the use of inductive reasoning and those psychological abilities that are common to creative thinkers.[1] Working with different numeration systems, or number bases, will afford the student the opportunity to see how a numeral acts in a different number system. By working with various number bases, a student can gain insight into the value of the decimal system.[2]

The student will be shown the following techniques: converting decimal numbers into a different numeration system; converting numbers of various number systems into the decimal system; performing addition, subtraction, multiplication, and division in numeration systems other than the decimal system; and solving an anecdote written in a different number system.

Congruence

"Although congruences form a vital tool in the theory of integers, Gauss recognized their utility, also, in showing certain polynomial equations to have no rational roots."[3] Congruence is often applied to happenings of a recurring nature, such as the recording of time.[4] The student will be made aware that modular mathematics is encountered everywhere in daily life from clocks to calendars.

One of the applications of congruences is the technique of casting out nines. Casting out nines is a useful method for checking addition and multiplication and can lend accuracy to a student's performance.[5] This property of the number nine has been known since ancient times and its discovery led to new questions and inquiries.[6] William B. Wetherbee postulates that the Romans were probably the first to use this process, followed by the Arabs, and eventually the Hindus in their work with astronomy. It was taught early in America's history, disappearing during the nineteenth century, only to return at the beginning of the twentieth century.[7] Philip Davis writes that "it remains today as a source of amusement, the basis of many number tricks involving large numbers, and a fine introduction to a part of number theory known as the Theory of Residues."[8]

Other techniques will include: solving linear congruences; applying the Chinese Remainder Theorem; finding remainders of large numbers; designing a round robin tournament; understanding ISBN numbers; and discovering the theories behind divisibility tests.

Diophantine Equations

J. A. H. Hunter and Joseph S. Madachy, describing Diophantos as the most famous Greek mathematician of his day, state that his methods for solving these types of problems "were centuries in advance of the general level of mathematical knowledge of those days."[9] Kenneth H. Rosen also makes a case for the learning of these equations. Solving Diophantine equations will enhance the student's algebraic skills as well as reinforce the student's ability to solve systems of equations.[10]

The student will study both linear and nonlinear Diophantine equations. The use of congruences, which will have been studied in the previous section, will facilitate the solving of linear Diophantine equations.

Nonlinear Diophantine equations will also be solved, and some of their applications will be examined. One such application includes the study of the Pythagorean triples, or finding solutions to the equation $x^2 + y^2 = z^2$. James Fey reports that Pythagoras was not the first person to study these triples; he states that the Babylonians also used them.[11] Knowledge of their inner workings was extremely helpful in the construction of buildings.[12] "These integers have, in modern times, led to many discoveries in number theory and also to many perplexing problems, some of which still await solutions."[13]

Other applications of nonlinear Diophantine equations will include integers which can be expressed as the sum of two squares ($x^2 + y^2 = z$) and the famous Pell equation.

<u>Fibonacci Numbers</u>

Jerome S. Meyer attests to the "fascinating features"[14] of these numbers. An understanding of this series will lead to a great comprehension of a certain aspect of botany--the arrangement of leaves on a stem.[15] There is also a connection between the Fibonacci sequence and the golden section as well as with other branches of mathematics, such as random numbers, primes, and factorization properties.[16]

<u>The Golden Section</u>

This has probably been known even before the time of the Greeks. The golden rectangle represents the aesthetic and artistic properties of this ratio.[17] The name phi, sometimes used to represent the golden section, was selected by an American who chose it because it represented the first name of the Greek who used the ratio in his sculpture.[18]

The student will learn that the golden ratio, or golden section, is the only number which is transformed into its own reciprocal by subtracting the number one.[19] The student will also study the occurrence of the golden section among the sides and diagonals of a regular polygon.

The Imaginary Numbers

The imaginary numbers are vital in their role with complex numbers and function theory. "Imaginaries are useful and essential to the development of mathematics and developed from the logical extension of certain processes."[20] "The imaginary number, i, plays a vital part in higher mathematics, physics and, particularly, theoretical electricity."[21] The core curriculum will then be reinforced by the high school student's exposure to imaginary numbers in a mathematics appreciation course.

The Exponential Function

Using nonalgebraic numbers, work with hyperbolic logarithms by such scientists as John Napier, John Speidell, James Gregory, Newton, and Leibniz led to the identification of these numbers.[22] Learning how to facilitate these computations may encourage students in their own searches for ways to utilize these numbers. Edward Kasner and James Newman allow that "one of the fruits of higher education is the illuminating view that a logarithm is merely a number that is found in a table. We shall have to widen the curriculum."[23] "Besides serving as the base for the natural logarithms, the exponential function, e, is a number useful everywhere in mathematics and applied science. No other mathematical constant, not even π, is more closely connected with human affairs than e. It has helped to do one thing better than any number yet discovered. It has played an integral part in helping mathematicians describe and predict what is for man the most important of all natural phenomena--that of growth."[24]

Pi

Kasner and Newman ask why so much time has been devoted to pi (π). One reason is to find a clue to its transcendental nature, while a second "the fact that π, a purely geometric ratio, could be evolved out of so many arithmetic relationships--out of infinite series, with apparently little or no relation to geometry--was a never-ending source of wonder and a never-ending stimulus to mathematical activity."[25] James K. Bidwell, in an article about Archimedes, reminds the reader that this great scientist, who described π, wrote mathematics in a style that is still very readable today.[26] The applications of π are so numerous that one can hardly doubt the value of studying the properties of π in a course on mathematics appreciation.

Perfect Numbers

"Perfect numbers are not useful in the construction of bombs. In fact, they are not useful at all. They are merely interesting, and their story is an interesting one."[27] There is much that remains to be discovered about the development of perfect numbers. Students will be encouraged to pursue conjectures not yet proven, such as the search for an odd perfect number.

Numbers with Shape

Fermat used the principles of these numbers with shape in the summation of certain series.[28] Some of these numbers of shape include triangular numbers, square numbers, and oblong numbers. Like perfect numbers, these numbers of shape contain many unusual properties which the student will be encouraged to pursue.

Ciphers

The history of the world changed because of the use of ciphers, or codes. Codes range from the very simple to the extremely complex and almost impossible to decode. Codes can use either letters or numerals.[29] Kenneth H. Rosen mentions the importance of ciphers with respect to number theory.[30] By working with ciphers the student will gain experience with prime numbers and deductive reasoning.

Magic Squares

These are probably of Chinese origin and still have a connection with mysticism in Asian countries. In Europe they were connected to alchemy and astrology. They have been applied to problems in probability and analysis and most recently in the design of experiments.[31] Agricultural research has benefitted from the application of certain magic squares, as has atomic research, marketing research, and sociology.[32] "Magic squares brilliantly reveal the intrinsic harmony and symmetry of numbers; with their curious and mystic charm they appear to betray some hidden intelligence that governs the cosmic order that dominates all existence. They have been compared to a mirror reflecting the symmetry of the universe, the harmonies of nature, the divine norm. It is not surprising that they have always exercised a great influence on thinking people."[33] "The beauty of magic squares is they can be used as simple recreations or they can be studied mathematically. They can find a place in the enjoyment of children as well as the mathematical inspections of adults."[34] Students in the mathematics appreciation course will practice both odd and even magic squares.

Root Extraction

Root extractions enable the student to better visualize binomial expansions. "It is worthwhile to understand the why and the wherefore of these operations.[35] In the proposed mathematics course, students will be practicing the solutions to square roots and cube roots without the use of a calculator.

Notes

[1] Aaron Bakst, Mathematics--Its Magic and Mystery (Toronto: D. Van Nostrand Company, Inc., 1967), 11.

[2] Irving Adler, A New Look at Arithmetic (New York: The John Day Company, 1964), 46-53.

[3] The National Council of Teachers of Mathematics, Historical Topics for the Mathematics Classroom (Washington, D.C.: The National Council of Teachers of Mathematics, Inc., 1969), 288.

[4] J. Richard Byrne, Number Systems: An Elementary Approach (New York: McGraw-Hill Book Company, 1967), 127.

[5] Jerome S, Meyer, Fun with Mathematics (Cleveland: The World Publishing Company, 1952), 57.

[6] Constance Reid, From Zero to Infinity (New York: Thomas Y. Crowell Company, 1960), 122.

[7] Historical Topics, 140.

[8] Philip Davis, The Lore of Large Numbers (New York: Random House, Inc.,1961), 94.

[9] J.A.H. Hunter and Joseph Madachy, Mathematical Diversions (New York: Dover Publications, Inc., 1975), 52.

[10] Kenneth H. Rosen, Elementary Number Theory and Its Applications (Reading, Massachusetts: Addison-Wesley Publishing Company, 1988), 104.

[11] Historical Topics, 68.

[12] Lancelot Hogben, Mathematics for the Million (New York: W.W. Norton and Company, Inc., 1968), 50.

[13] Tobias Dantzig, Number: The Language of Science (New York: The Macmillan Company, 1967), 285.

[14] Meyer, 65.

[15] Hunter and Madachy, 20.

[16] Historical Topics, 79.

[17] Historical Topics, 206.

[18] Martin Gardner, Mathematical Puzzles and Diversions (New York: Simon and Schuster, 1961), 91.

[19] Hunter and Madacachy, 15.

[20] Edward Kasner and James Newman, Mathematics and the Imagination (New York: Simon and Schuster, 1960), 92.

[21] Meyer, 90.

[22] Historical Topics, 83, 154.

[23] Kasner and Newman, 83.

[24] Kasner and Newman, 84.

[25] Kasner, 78.

[26] James K. Bidwell, "Humanize Your Classroom with the History of Mathematics," Mathematics Teacher, Sept. 1993, 127.

[27] Reid, 84.

[28] Historical Topics, 57.

[29] Bakst, 79-80.

[30] Rosen, 208.

[31] Historical Topics, 80-81.

[32] Hunter, 34.

[33] Jim Moran, The Wonders of Magic Squares (New York: Vintage Books, 1981), 6.

[34] Karen Dee Michalowicz, "The Magic Square: More Than a Mathematical Recreation," School and Science Mathematics, Jan. 1995, 43.

[35] Geoffrey Mott-Smith, Mathematical Puzzles for Beginners and Enthusiasts (Philadelphia: The Blakiston Company, 1946), 236-239.

CHAPTER 4

A Course in Mathematics Appreciation

Introduction

Mathematics appreciation is an elective course designed for the 11th or 12th grade high school student who has an interest in mathematics. The student must have successfully completed courses in algebra I, algebra II, and geometry in order to enroll in the mathematics appreciation course. In addition, it is highly recommended that the student be concurrently enrolled in a trigonometry or calculus course, depending on the student's grade level. The mathematics appreciation course is designed to be nine weeks in length.

The Topics

There are thirteen topics, with each topic providing the student with several days of mathematical investigation. The classroom teacher is encouraged to assign problems to the class according to the ability and interest level of the class.

Numeration Systems

Introduction

Various civilizations have used number systems other than the decimal system (base ten), such as the base sixty system of the Babylonians or the base twenty system of the Mayan Indians.[1] The numbers will be written with subscripts representing the base, so that 35_8 will be read as "thirty-five base eight".

Counting

In the octal system (base eight), only the digits zero through seven are used. Counting from one to one-hundred in base eight is done as follows: one, two, . . . , six, seven, ten, eleven, . . . , sixteen, seventeen, twenty, twenty-one, . . . , twenty-six, twenty-seven, thirty, . . . , seventy, seventy-one, . . . , seventy-six, seventy-seven, and finally, one-hundred. Since there is no eights digit in base eight, the number seven is followed by ten, and the number seventy-seven is followed by one-hundred. Notice that there are eight (8^1) integers from one through ten in base eight, and that there are sixty-four (8^2) integers from one through one-hundred in base eight.

"If we had twelve fingers instead of ten, we would tend to count objects in groups of twelve."[2] It could also be argued that a base twelve number system would be better than a base ten number system, since twelve has more divisors than ten. Since twelve digits are required in base twelve, two additional digits T (called dek) and E (called el) must be included. Counting from one to one-hundred would be performed as follows: one, two, . . . , eight, nine, dek, el, twenty, twenty-one, . . . , twenty-eight, twenty-nine, twenty-dek, twenty-el, thirty, . . . , ninety, ninety-one, . . . , ninety-eight, ninety-nine, ninety-dek, ninety-el, dekty, dekty-one, . . . , dekty-nine, dekty-dek, dekty-el, elty, elty-one, . . . , elty-nine, elty-dek, elty-el, and finally, one-hundred. Notice that there are a total of one-hundred and forty-four (12^2) numbers from one through one-hundred in base twelve.

Converting Between Number Bases

A base eight number is converted to a decimal number by using expanded notation in powers of eight.

Problem: Convert 254_8 to the decimal system.

Solution:

$$\begin{aligned} 254_8 &= (2 \times 8^2) + (5 \times 8^1) + (4 \times 8^0) \\ &= (2 \times 64) + (5 \times 8) + (4 \times 1) \\ &= 128 + 40 + 4 \\ &= 172_{10}. \end{aligned}$$

Therefore, $254_8 = 172_{10}$.

To convert from decimal to octal, find the highest power of eight which is not greater than the decimal number, and then divide that highest power of eight into the decimal number. The quotient is the first digit of the octal number. Divide the next highest power of eight into the remainder. This will yield the next digit of the octal. Repeat the process until eight to the zero power is the last divisor to be used.

Problem: Convert 172_{10} into the base eight number system.

Solution: Since $8^3 = 512$ is larger than 172, 8^2 is used for the first division.

$$172 / 64 = 2 \quad \text{(remainder 44)},$$
$$44 / 8 = 5 \quad \text{(remainder 4)},$$
$$4 / 1 = 4 \quad \text{(remainder 0)}.$$

Therefore, $172_{10} = 254_8$.

The octal system (base eight) is useful because of the ease of conversion between it and the binary system (base two). The binary system is used by computers since that system consists of the digits zero and one, and all numbers can be represented electronically by switches where one is "on" and zero is "off".[3] Counting from one to one-hundred would be done as follows: one, ten, eleven, and one-hundred. Note that there are two-squared or four numbers from one to one-hundred in the binary system. Conversions between base two numbers and decimal numbers are accomplished by using the same methods as outline above.

Problem: Convert 11001_2 into decimal form.

Solution:

$$\begin{aligned} 11001_2 &= (1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) \\ &= (1 \times 16) + (1 \times 8) + (0 \times 4) + (0 \times 2) + (1 \times 1) \\ &= 16 + 8 + 0 + 0 + 1 \\ &= 25_{10}. \end{aligned}$$

Therefore, $11001_2 = 25_{10}$.

Problem: Convert 25_{10} into binary form.

Solution:

$$25 / 16 = 1 \quad \text{(remainder 9);}$$
$$9 / 8 = 1 \quad \text{(remainder 1);}$$
$$1 / 4 = 0 \quad \text{(remainder 1);}$$
$$1 / 2 = 0 \quad \text{(remainder 1);}$$
$$1 / 1 = 1 \quad \text{(remainder 0).}$$

Therefore, $25_{10} = 11001_2$.

Problem: Convert $3TE_{12}$ into decimal form.

Solution:

$$\begin{aligned} 3TE_{12} &= (3 \text{ x } 12^2) + (T \text{ x } 12^1) + (E \text{ x } 12^0) \\ &= (3 \text{ x } 144) + (10 \text{ x } 12) + (11 \text{ x } 1) \\ &= 432 + 120 + 11 \\ &= 563_{10}. \end{aligned}$$

Therefore, $3TE_{12} = 563_{10}$.

Problem: Convert 563_{10} into base twelve.

Solution:

$$563 / 144 = 3 \quad \text{(remainder 131),}$$
$$131 / 12 = T \quad \text{(remainder 11),}$$
$$11 / 1 = E \quad \text{(remainder 0).}$$

And so, $563_{10} = 3TE_{12}$.

The hexadecimal (base sixteen) system is in computers because of its relationship of sixteen (two to the fourth power) to the number two. Since the hexadecimal system uses sixteen digits, zero through fifteen, computer programmers commonly use the letters A, B,C, D, E, and F as the extra digits.[4]

Two examples of the ease of conversion between the hexadecimal and binary systems are based on the principle that one hexadecimal digit is equal to four binary digits.

Problem: Convert $4AE_{16}$ to binary.

Solution:

$$4 = 0100,$$
$$A = 1010,$$
$$E = 1110.$$

So, $4AE_{16} = 10010101110_2$.

Problem: Convert 11101101110_2 to hexadecimal.

Solution: Break the binary into groups of four starting at the right.

0111 = 7,
0110 = 6,
1110 = E.

Thus, $11101101110_2 = 76E_{16}$.

Simple Operations in Other Bases

The main reason for studying numeration systems in bases other than ten is not to learn arbitrary numeration systems, but rather to gain insight into the structure of the familiar Hindu-Arabic decimal system.[5]

Addition in any numerical system depends on place value as well as face value. When a sum has two digits, the left digit is "carried" to the next position. An addition table is helpful although the serious student should be able to perform addition without it.

TABLE 1

Addition in Base Five

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Problem: Add 433_5 and 214_5.

Solution:

Step 1:
$$\begin{array}{r} 4\,3\,3 \\ +\underline{2\,1\,4} \\ 2 \end{array}$$
$3 + 4 = 12_5$; write 2 and carry 1.

Step 2:
$$\begin{array}{r} {}^{1} \\ 4\,3\,3 \\ +\underline{2\,1\,4} \\ 0\,2 \end{array}$$
$1 + 3 + 1 = 10_5$; write 0 and carry 1.

Step 3:
$$\begin{array}{r} {}^{1}\quad\; \\ 4\,3\,3 \\ +\underline{2\,1\,4} \\ 1\,2\,0\,2 \end{array}$$
$1 + 4 + 2 = 12_5$; write 12.

Therefore, $433_5 + 214_5 = 1202_5$.

Subtraction is basically the same in base five as in the decimal system. Sometimes "borrowing" may be required.

Problem: Subtract 214_5 from 1202_5.

Solution:

```
            1 4 1
Step 1:   1 2 0 2       12 - 4 = 3₅ (See addition table).
          - 2 1 4        Borrow 1 from the 2 leaving 1,
                3        borrow 1 from the 10 leaving 4, and write 3.
```

```
            1 4 1
Step 2:   1 2 0 2       4 - 1 = 3₅ (See addition table). Write 3.
          - 2 1 4
              3 3
```

```
            1
Step 3:   1 2 0 2       11 - 2 = 4₅ (See addition table). Write 4.
          - 2 1 4
            4 3 3
```

Therefore, $1202_5 - 214_5 = 433_5$.

Multiplication and division in base five are made easier by the use of a multiplication table. Multiplication in base five may require "carrying over". See Table 2 below showing base five multiplication.

TABLE 2

Multiplication in Base Five

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Problem: Multiply 34_5 times 42_5.

Solution:

```
             1
Step 1:      3 4    2 x 4 = 13₅. Write 3, carry 1.
           x 4 2
               3
```

Step 2:

```
    1
    3 4
  x 4 2
  -----
  1 2 3
```

$2 \times 3 + 1 = 12_5$. Write 12.

Step 3:

```
    3 4
  x 4 2
  -----
  1 2 3
    1
```

$4 \times 4 = 31_5$. Write 1, carry 3.

Step 4:

```
    3
    3 4
  x 4 2
  -----
  1 2 3
3 0 1
-------
```

$4 \times 3 + 3 = 30_5$. Write 30.

Step 5:

```
    3 4
  x 4 2
  -----
  1 2 3
3 0 1
-------
3 1 3 3
```

Add.

Therefore: $34_5 \times 42_5 = 3133_5$.

Problem: Divide 4232_5 by 3_5.

Solution: The reasoning in base five is the same as in base ten.

Step 1: 3 divides into 4 once.
Multiply and divide.

```
    1
3 | 4 2 3 2
    3
    -
    1
```

Step 2: Bring down the 2.
3 divides into 12 twice.
Multiply and subtract.

```
    1 2
3 | 4 2 3 2
    3
    -
    1 2
    1 1
    ---
      1
```

Step 3: Bring down the 3.
3 divides into 13 twice.
Multiply and subtract.

```
    1 2 2
3 | 4 2 3 2
    3
    -
    1 2
    1 1
    -----
      1 3
      1 1
      ---
        2
```

Step 4: Bring down the 2.
3 divides into 22 four times.
Multiply and subtract.

```
      1 2 2 4
  3 | 4 2 3 2
      3
      1 2
      1 1
        1 3
        1 1
          2 2
          2 2
            0
```

Since the remainder is zero: $4232_5 / 3_5 = 1224_5$.

A Base Four Story[6]

An eccentric mathematician, when he died, left a stack of unpublished papers. When his friends were sorting them, they came across the following statement:

"I graduated from college when I was 44 years old. A year later, I, a 100-year-old man, married a 34-year-old young girl. Since the difference in our ages was only 11 years, we had many common interests and hopes. A few years later we had a family of 10 children. I had a college job, and my salary was $1300 a month. One-tenth of my salary went for the support of my parents. However, the balance of $1,120 was more than sufficient for us to live on comfortably."

How eccentric was the mathematician? The student is encouraged to rewrite the puzzler before reading the explanation below.

Note that when 1 was added to 44, the result was 100. Since 44 is the highest two-digit number, the eccentric mathematician must have been using a base five numeration system. The base five numbers can therefore be converted to base ten numbers as follows:

$$44_5 = 4 \text{ x } 5^1 + 4 \text{ x } 5^0 = 24_{10},$$
$$100_5 = 1 \text{ x } 5^2 + 0 \text{ x } 5^1 + 0 \text{ x } 5^0 = 25_{10},$$
$$34_5 = 3 \text{ x } 5^1 + 4 \text{ x } 5^0 = 19_{10},$$
$$11_5 = 1 \text{ x } 5^1 + 1 \text{ x } 5^0 = 6_{10},$$
$$10_5 = 1 \text{ x } 5^1 + 0 \text{ x } 5^0 = 5_{10},$$
$$1300_5 = 1 \text{ x } 5^3 + 3 \text{ x } 5^2 + 0 \text{ x } 5^1 + 0 \text{ x } 5^0 = 200_{10},$$
$$(1/10)_5 = 1/(1 \text{ x } 5^1 + 0 \text{ x } 5^0) = 1/5,$$
$$1120_5 = 1 \text{ x } 5^3 + 1 \text{ x } 5^2 + 2 \text{ x } 5^1 + 0 \text{ x } 5^0 = 160_{10}.$$

"I graduated from college when I was 24 years old. A year later, I, a 25-year-old man, married a 19-year-old young girl. Since the difference in our ages was only 6 years, we had many common interests and hopes. A few years later we had a family of 5 children. I had a college job, and my salary was $200 a month. One-fifth of my salary went for the support of my parents. However, the balance of $160 was more than sufficient for us to live on comfortably."

The student is encouraged to write an unusual anecdote in a numeration system other than base five or base ten.

<u>A Base Two Trick</u>[7]

Place nine small envelopes and $5.11 in change on a table. Distribute the money in the envelopes and then announce that you can hand over any sum of money up to $5.11 without counting the money. Someone names $3.46 and you hand that person certain envelopes. The person counts the money and finds that

1 envelope contains	$2.56,
1 envelope contains	$0.64,
1 envelope contains	$0.16,
1 envelope contains	$0.08,
1 envelope contains	<u>$0.02.</u>
Total	$3.46.

Again, the student is encouraged to solve this anecdote before reading on.

The problem is easily solved using the base two numeration system. Recognize that 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 511 and that each of these powers of two corresponds to a dollar amount distributed among the nine envelopes. Arrange the envelopes in the following order.

1st	2nd	3rd	4th	5th	6th	7th	8th	9th
2.56	1.28	0.64	0.32	0.16	0.08	0.04	0.02	0.01

To choose which group of envelopes contains exactly $3.46, simply convert 346 mentally from the decimal system to the binary system using the methods previously discussed. Since, $346_{10} = 101, 011, 010_2$, choose those envelopes which correspond to the ones in the binary number. Thus, the performer of the trick chooses envelopes 1, 3, 5, 6, and 8 whose sum is equal to $3.46.

Conclusion

Numeration systems have been examined from an historical point of view by recognizing the different number systems used by different cultures such as the Babylonians. The anecdotes given above allow the student to apply various number systems in a recreational way. Finally, converting from one base to another and performing simple operations in bases other than base ten, affords the student the opportunity to make practical use of the place value system inherent in all of the numeration systems investigated in this chapter thereby increasing the student's understanding of the base ten number system.

Congruence

Introduction

The special language of congruences is extremely useful in number theory. The language of congruences was introduced by Karl Friedrich Gauss in 1801, when he was twenty-four years old.[8]

Congruences often arise in everyday life. For example, clocks work on modulo 12 or 24 when measuring hours and modulo 60 when measuring minutes and seconds. Calendars work on modulo 7 when measuring weeks and modulo 12 when measuring months.[9]

A set of integers can be divided into m different sets called congruence classes modulo m if all the members of a particular class produce the same remainder when divided by m. Thus, $7 \equiv 12 \pmod 5$, which reads seven is congruent to twelve in modulo five, means that 7 and 12 are in the same congruence class modulo 5 since both yield a remainder of 2 when divided by 5.

The five congruence classes modulo 5 are given by:

$$\ldots \equiv -10 \equiv -5 \equiv 0 \equiv 5 \equiv 10 \equiv \ldots \pmod 5$$
$$\ldots \equiv -9 \equiv -4 \equiv 1 \equiv 6 \equiv 11 \equiv \ldots \pmod 5$$
$$\ldots \equiv -8 \equiv -3 \equiv 2 \equiv 7 \equiv 12 \equiv \ldots \pmod 5$$
$$\ldots \equiv -7 \equiv -2 \equiv 3 \equiv 8 \equiv 13 \equiv \ldots \pmod 5$$
$$\ldots \equiv -6 \equiv -1 \equiv 4 \equiv 9 \equiv 14 \equiv \ldots \pmod 5$$

Problem: Is $23 \equiv 48 \pmod 5$?

Solution: Dividing both 23 and 48 by 5 yields a remainder of 3.

Therefore, $23 \equiv 48 \pmod 5$.

Problem: Add $(9 + 13) \pmod 5$.

Solution: First add 9 + 13 to get 22. But 22/5 leaves a remainder of 2.

Therefore, $9 + 13 \equiv 2 \pmod 5$.

Problem: Multiply $(8 \times 7) \pmod 5$.

Solution: Multiply 8 x 7 to get 56. But 56/5 leaves a remainder of 1.

Therefore, $8 \times 7 \equiv 1 \pmod 5$.

The following three theorems on congruence will be stated without proof. The proofs, which can be found in most books on number theory, should be attempted by the student. These theorems will prove valuable in the remainder of the congruence section.

If a, b, c, k, and m are integers such that $k > 0$ and $m > 0$, then:

Theorem 1: If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then, $(a + c) \equiv (b + d) \pmod m$.

Theorem 2: If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then, $(a \times c) \equiv (b \times d) \pmod m$.

Theorem 3: If $a \equiv b \pmod m$, then $a^k \equiv b^k \pmod m$.

Linear Congruences

A congruence of the form $ax \equiv b \pmod m$ is called a linear congruence in one variable.[10] Asking how many of the m congruence classes are solutions to a linear equation is the same as asking how many incongruent solutions there are in modulo m. Also, the greatest common divisor of a and b is written (a, b). Let a, b, and m be integers with $m > 0$ and $(a, m) = d$. Then, if $d \nmid b$, then $ax \equiv b \pmod{m}$ has no solutions. And if $d \mid b$, then $ax \equiv b \pmod{m}$ has exactly d incongruent solutions modulo m. Below are four examples which illustrate various methods used in solving linear congruencies.

Problem: Solve the equation $5x \equiv 4 \pmod{9}$.

Solution: Note that $(5, 9) = 1$ so that there is a unique solution modulo 9. The equation is solved by multiplying both sides of the equation by $\overline{a}$, the multiplicative inverse (mod m) of a. Thus, $a \times \overline{a}$ will be equal to 1, and the equation will be solved. In this case, the inverse of 5 is 2 since $2 \times 5 \equiv 1 \pmod{9}$. Multiplying both sides of the equation by 2 yields,

$$5x \equiv 4 \pmod{9},$$
$$(2)(5x) \equiv (2)(4) \pmod{9}.$$

Therefore, $x \equiv 8 \pmod{9}$.

Problem: Solve the equation $36x \equiv 8 \pmod{102}$.

Solution: Note that $(36, 102) = 6 \nmid 8$.

Therefore, there is no solution (mod102).

Problem: Solve the equation $6x \equiv 15 \pmod{21}$.

Solution: Note that $(6, 21) = 3 \mid 15$. Thus there are 3 solutions. The process is simplified by reducing $6x \equiv 15 \pmod{21}$ to the form $2x \equiv 5 \pmod{7}$. The particular solution is then found by multiplying both sides of the equation by the inverse of 2 modulo 7. Thus,

$$2x \equiv 5 \pmod{7},$$
$$(4)(2x) \equiv (4)(5) \pmod{7},$$
$$x \equiv 20 \equiv 6 \pmod{7}.$$

The general solution is found by listing all possible solutions in modulo class 21. Thus, $x = 6 + 7T$ where $T = 0, 1$, and 2.

Therefore, all solutions are $x \equiv 6$, 13, and 20 (mod 21).

Problem: Solve the equation $25x \equiv 15 \pmod{29}$.

Solution: Note that $(25, 29) = 1$ indicating that there is exactly one solution. However, it is difficult to find the inverse of 25 (mod 29) by trial and error. Fortunately, the division algorithm will simplify the task. Dividing 29 by 25 yields a remainder of 4. Then, using the division algorithm yields, $4 = 29 - 1(25)$. Dividing 25 by 4 yields a remainder of 1. Therefore, by the division algorithm, $1 = 25 - 6(4)$.

Thus,
$$1 = 25 - 6(4),$$
$$1 = 25 - 6[(29 - 1(25)],$$
$$1 = 7(25) - 6(29).$$

But,
$$1 = 7(25) - 6(29),$$
$$1 \equiv 7(25) - 6(0) \pmod{29},$$
$$1 \equiv 7(25) \pmod{29}.$$

Thus, 7 is the inverse of 25 modulo 29.

Hence,
$$25x \equiv 15 \pmod{29},$$
$$(7)(25x) \equiv (7)(15) \pmod{29},$$
$$1x \equiv 105 \pmod{29}.$$

Therefore, $x \equiv 18 \pmod{29}$.

The student should create, and then solve, several linear congruences.

<u>The Chinese Remainder Theorem (Simultaneous Congruences)</u>

The Chinese Remainder Theorem deals with simultaneous linear congruences in one variable, with different moduli. Such systems arose in ancient Chinese puzzles.[11] The method for solving such puzzles is given in the following theorem:

Theorem 4: Let $m_1, m_2, \ldots, m_k$ be pairwise relatively prime positive integers.

Then the system of congruence

$$x \equiv a_1 \pmod{m_1},$$
$$x \equiv a_2 \pmod{m_2},$$
$$\vdots$$
$$x \equiv a_k \pmod{m_k},$$

has a unique solution modulo $M = m_1m_2 \ldots m_k$.

Problem: Find the smallest number that leaves a remainder of 2 when divided by 5, a remainder of 3 when divided by 7, and a remainder of 4 when divided by 11.

Solution: The system of congruences which represents the puzzle is:

$x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$, and $x \equiv 4 \pmod{11}$.

$M = (m_1)(m_2)(m_3) = (5)(7)(11) = 385$.

The technique is to find:

$2 \pmod 5 \equiv (?)(7)(11) \pmod 5 \equiv (1)(7)(11) \pmod 5 \equiv 77 \pmod 5$;

$3 \pmod 7 \equiv (?)(5)(11) \pmod 7 \equiv (4)(5)(11) \pmod 7 \equiv 220 \pmod 7$;

$4 \pmod{11} \equiv (?)(5)(7) \pmod{11} \equiv (2)(5)(7) \pmod{11} \equiv 70 \pmod{11}$.

Thus, $x \equiv 77 + 220 + 70 \equiv 367 + 385\,T$.

Therefore, if $T = 0$, then, $x \equiv 367 \pmod{385}$.

Problem: Find a multiple of 11 that leaves a remainder of 1 when divided by 2, 3, 5, and 7.

Solution: The system of congruences which represents the puzzle is:

$x \equiv 1 \pmod 2$, $x \equiv 1 \pmod 3$, $x \equiv 1 \pmod 5$, $x \equiv 1 \pmod 7$, and $x \equiv 0 \pmod{11}$.

$M = (m_1)(m_2)(m_3)(m_4)(m_5) = (2)(3)(5)(7)(11) = 2310$.

The technique is to find:

$1 \pmod 2 \equiv (?)(3)(5)(7)(11) \pmod 2 \equiv (1)(3)(5)(7)(11) \pmod 2 \equiv$
$1155 \pmod 2$;

$1 \pmod 3 \equiv (?)(2)(5)(7)(11) \pmod 3 \equiv (2)(2)(5)(7)(11) \pmod 3 \equiv$
$1540 \pmod 3$;

$1 \pmod 5 \equiv (?)(2)(3)(7)(11) \pmod 5 \equiv (3)(2)(3)(7)(11) \pmod 5 \equiv$
$1386 \pmod 5$;

$1 \pmod 7 \equiv (?)(2)(3)(5)(11) \pmod 7 \equiv (1)(2)(3)(5)(11) \pmod 7 \equiv$
$330 \pmod 7$;

$0 \pmod{11}$.

Since, $x \equiv 1155 + 1540 + 1386 + 330 + 0 \equiv 4411 + 2310T$.

Therefore, if $T = -1$, then, $x \equiv 2101 \pmod{2310}$.

The student should try the following problem: Five men and a monkey are shipwrecked on an island. The men have collected a pile of coconuts which they plan to divide equally among themselves the next morning. Not trusting the other men, one of the group wakes up during the night and divides the coconuts into five equal parts with one left over, which he gives to the monkey. He then hides his portion of the pile. During the night, each of the other four men does exactly the same thing by dividing the pile they find into five equal parts, leaving one coconut for the monkey and hiding his portion. In the morning, the men gather and split the remaining pile of coconuts, leaving one for the monkey. What is the minimum number of coconuts that the men could have collected for their original pile?[12]

<u>Remainders of Large Numbers</u>

Problem: Find the remainder when 2^{102} is divided by 25.

Solution: It is seen that $2^{10} = 1024 \equiv -1 \pmod{25}$.

$$\begin{aligned}\text{Then, } 2^{102} &= (2^{100})(2^2)\\ &= (2^{10})^{10}(4)\end{aligned}$$

$$\equiv (-1)^{10}(4)(\text{mod } 25)$$
$$\equiv (1)(4)(\text{mod } 25)$$
$$\equiv 4 \ (\text{mod } 25).$$

Thus, the remainder is equal to 4.

Problem: Find the remainder when 41^{65} is divided by 7.

Solution: It is easily seen that $41 \equiv -1 \ (\text{mod } 7)$.

$$\text{Therefore, } 41^{65} \equiv (-1)^{65} \ (\text{mod } 7)$$
$$\equiv (-1) \ (\text{mod } 7)$$
$$\equiv 6 \ (\text{mod } 7).$$

Thus, the remainder is equal to 6.

<u>ISBN Numbers</u>

An ISBN is a ten-digit number; the ISBN number for one of the references in this study is 0-673-38829-8.[13] The first digit, in this case 0, identifies the book as having been published in an English-speaking country. The next digits, 673, identify the publisher, while 38829 identify the particular book. The final digit, 8, is the check digit. To find the check digit, start at the left of the ISBN number and multiply the digits by 10, 9, 8, . . . , 4, 3, and 2, respectively, and then add these products. In this example, $10(0) + 9(6) + 8(7) + 7(3) + 6(3) + 5(8) + 4(8) + 3(2) + 2(9) = 245$. The check digit is the smallest digit that must be added to the result, 245, so that the final sum is congruent to 0 modulo 11. In this case, $245 + 8 = 253 \equiv 0 \ (\text{mod } 11)$. If the required check number is 10, the letter X is used instead of ten.

When the order for a book is received, the ISBN is entered into a computer, and the check number is evaluated. If this result does not match the check number on the order, then the order will not be processed.

The student should verify the ISBN from an assortment of available books.

Round-Robin Tournaments

Congruences can be used in order to schedule round-robin tournaments, where N different teams play every other team exactly once. The method about to be described was developed by J. E. Freund.[14]

Note that if N is odd, then not all teams can play every round. In that case, a dummy team is added to make N even. A team that is scheduled to play the dummy team is then said to have drawn a bye.

Label the N teams with the integers 1, 2, 3, . . . , N-1, and N. Pairings in the kth round are scheduled in the following way. Team i is paired with team j, where $i \neq N$, $j \neq N$, $i \neq j$, and $i + j \equiv k \pmod{N - 1}$. This will schedule all teams in round k except for team N and the one team i for which $2i \equiv k \pmod{N - 1}$. These two teams will then be matched with each other in round k, thus completing the pairings.

For example, for N = 5, the pairings are listed in Table 3 below.

TABLE 3

Pairings for a Five-Team Round-Robin Tournament

Team / Round	1	2	3	4	5
1	5	4	bye	2	1
2	bye	5	4	3	2
3	2	1	5	bye	3
4	3	bye	1	5	4
5	4	3	2	1	bye

The student should set up a round-robin tournament schedule for 6, 7, 8, 9, and 10 teams.

Casting Out Nines

There is one property of the number nine, known since antiquity, that does not depend upon its relationship with other numbers. This is the fact that nine, divided into any power of ten, always leaves a remainder of one. In the days when computations were performed on counting boards, nine was commonly used as a check. This ancient computational check was called casting out nines. An understanding of casting out nines requires a knowledge of the concepts of congruence and digital roots.[15]

The digital root of a number is the single integer reached by continued summation of the digits of the number.[16] Given the number 789, the digital root is found by the following process: the sum of the digits of 789 is 24; the sum of the digits of 24 is 6; thus the digital root of 789 is 6.

Note that if the digit "9" is not included in the digital root process, the digital root can still be found. Ignoring the "9" digit in 789, the sum of the digits of 78 is 15; the sum of the digits of 15 is 6; thus the same digital root is reached.

In a different example, consider the number 34567. Following the same process, the sum of the digits of 34567 is 25 and the sum of the digits of 25 is 7, which is the digital root.

Note, however, that the sum of 3 and 6 is 9, and that the sum 4 and 5 is also 9. Striking out the digits that sum to 9 (~~3456~~7) leaves 7 which, as just seen, is the digital root of 34567.

A question the student may ask is, "Can the digital root always be found by striking out either the nines digit(s) and/or those digits which sum to nine, and then performing continued summation on the remaining digits, thereby simplifying the calculations?"

The answer is yes, because if there are n such combinations of nine as previously described, and if the remaining digits sum to k, then:

$$\begin{aligned} 9\,n + k &= (10 - 1)\,n + k \\ &= 10\,n - 1\,n + k \\ &\equiv n - n + k \pmod{10} \\ &\equiv k \pmod{10}, \end{aligned}$$

and therefore, the nines can be "casted out".[17]

Another way to cast out nines is to subtract 9 repeatedly from a number until a new whole number less than 9 is left. Since division is repeated subtraction, to cast out nines from a number in this way means to divide the number by 9, and behold the remainder. The remainder is called the "excess of nines".[18]

"Why is finding the digital root of a number equivalent to finding the excess of nines?" This question is answered by noting that if n is any natural number, then $10^n \equiv 1 \pmod 9$. This can be proven by making use of Theorem 2:

$$\begin{aligned} &10 \equiv 1 \pmod 9 && \text{since } 10/9 \text{ yields a remainder of } 1; \\ &10^1 \equiv 1 \pmod 9 && \text{since } 10 = 10^1; \\ &10^2 \equiv 1 \pmod 9 && \text{by the multiplication property;} \\ &10^n \equiv 1 \pmod 9 && \text{by repeated use of the multiplication property.} \end{aligned}$$

For example, the number 526 has a digital root of 4. The excess of nines of 526 should also be equal to 4. Using the fact that $10^n \equiv 1 \pmod 9$, the excess of nines can be found in the following way:

$$\begin{aligned} 5 \times 100 &= 5 \times 10^2 \equiv 5 \times 1 \pmod 9 \equiv 5 \pmod 9; \\ 2 \times 10 &= 2 \times 10^1 \equiv 2 \times 1 \pmod 9 \equiv 2 \pmod 9; \\ 6 \times 1 &= 6 \times 10^0 \equiv 6 \times 1 \pmod 9 \equiv 6 \pmod 9. \end{aligned}$$

Therefore, $500 + 20 + 6 \equiv (5 + 2 + 6) \pmod 9 \equiv 4 \pmod 9$ which is the desired result.[19]

Casting out nines is often useful in checking addition problems. The reason this works is easy to demonstrate by making use of Theorem 1.

Let a, b, and c be natural numbers, and let a', b', and c' be their respective remainders, modulo 9. Since $a \equiv a'$ (mod 9), $b \equiv b'$ (mod 9), and $c \equiv c'$ (mod 9), then by the additive property, **if a + b = c, then a' + b' = c'.**[20]

The following addition problem illustrates an example of checking addition by casting out nines.

	a = 569		a' ≡ 2 (mod 9)
+	b = 273	+	b' ≡ 3 (mod 9)
	c = 842		c' ≡ 5 (mod 9) √

Casting out nines is also useful in checking multiplication problems. This is also easy to demonstrate by making use of Theorem 2.

Let a, b, and c be natural numbers, and let a', b', and c' be their respective remainders, modulo 9. Since $a \equiv a'$ (mod 9), $b \equiv b'$ (mod 9), and $c \equiv c'$ (mod 9), then by the multiplicative property, **if a x b = c, then a' x b' = c'.**[21]

The following multiplication problem illustrates an example of checking multiplication by casting out nines.

	a = 246		a' ≡ 3 (mod 9)
x	b = 53	x	b' ≡ 8 (mod 9)
	c = 13038		c' ≡ 6 (mod 9) √

In all numeration systems, checking is accomplished by casting out the highest digit in the system. Casting out sevens would be required in an octel numeration system, while casting out els would be required in a base twelve numeration system.

Check the following addition problem in base 5 by casting out **fours**:

	a = 434_5		a' ≡ 3 (mod 4)
+	b = 312_5	+	b' ≡ 2 (mod 4)
	c = 1301_5		c' ≡ 1 (mod 4) √

Casting out nines can lead to a better understanding of the decimal number system. Therefore, the student is encouraged to check addition and multiplication problems whenever the opportunity arises. Subtraction and division can be similarly checked by reversing the process.

Divisibility Tests[22]

Congruences can be used to devise divisibility tests for various integers based on their expansions with respect to different bases. Let the number $n = (a_k a_{k-1} \ldots a_1 a_0)_{10}$. Then the decimal expansion of n becomes $n = a_k 10^k + a_{k-1}10^{k-1} + \ldots + a_1 10^1 + a_0 10^0$.

The first test to be developed is for divisibility by 2. Since $10^1 \equiv 0 \pmod 2$, it follows that $10^2 \equiv 0 \pmod 2$, $10^3 \equiv 0 \pmod 2$, . . . , $10^k \equiv 0 \pmod 2$, for all positive integers k.

Then,

$$\begin{aligned} n &= a_k 10^k + a_{k-1}10^{k-1} + \ldots + a_2 10^2 + a_1 10^1 + a_0 10^0 \\ &\equiv a_k(0) + a_{k-1}(0) + \ldots + a_2(0) + a_1(0) + a_0(1) \\ &\equiv a_0 \pmod 2. \end{aligned}$$

Therefore, the digits a_k, a_{k-1}, . . . , a_2, and a_1 are "unimportant" when considering divisibility by two, since these digits are eliminated by their respective powers of ten. However, a_0 is "unprotected" by its respective power of ten since 10^0 is not zero in modulo two. Consequently, only the last digit needs to be tested to determine if n is divisible by 2.

Next, consider divisibility by 4. Since $10^2 \equiv 0 \pmod 4$, it follows that $10^3 \equiv 0 \pmod 4$, $10^4 \equiv 0 \pmod 4$, . . . , $10^k \equiv 0 \pmod 4$, for all positive integers k. Then,

$$\begin{aligned} n &= a_k 10^k + a_{k-1}10^{k-1} + \ldots + a_2 10^2 + a_1 10^1 + a_0 10^0 \\ &\equiv a_k(0) + a_{k-1}(0) + \ldots + a_2(0) + a_1(10) + a_0(1) \\ &\equiv a_1 a_0 \pmod 4. \end{aligned}$$

Therefore, the digits a_k, a_{k-1}, . . . , a_3, and a_2 are "unimportant" when considering divisibility by four, since these digits are eliminated by their respective powers of ten. However, a_1 and a_0 are "unprotected" by their respective powers of ten since 10^1 and 10^0 are not zero in modulo four. Hence, only the last two digits need to be tested to determine if n is divisible by 4.

In general, consider divisibility by 2^j. Since $10^1 \equiv 0 \pmod{2^1}$ and $10^2 \equiv 0 \pmod{2^2}$, it follows that $10^j \equiv 0 \pmod{2^j}$ for all positive integers j. Since$10^j \equiv 0 \pmod{2^j}$, it follows that $10^{j+1} \equiv 0 \pmod{2^j}$, $10^{j+2} \equiv 0 \pmod{2^j}$, . . . , $10^k \equiv 0 \pmod{2^j}$, for all positive integers k and j. Then,

$$
\begin{aligned}
n &= a_k 10^k + \ldots + a_j 10^j + a_{j-1} 10^{j-1} + \ldots + a_2 10^2 + a_1 10^1 + a_0 10 \\
&\equiv a_k(0) + \ldots + a_j(0) + a_{j-1} 10^{j-1} + \ldots + a_2(100) + a_1(10) + a_0(1) \\
&\equiv a_{j-1} a_{j-2} \ldots a_2 a_1 a_0 \pmod{2^j}.
\end{aligned}
$$

Therefore, the digits a_k, a_{k-1}, . . . a_j , are "unimportant" when considering divisibility by 2^j, since these digits are eliminated by their respective powers of ten. However, a_{j-1}, a_{j-2}, . . . , a_2, a_1, and a_0 are "unprotected" by their respective powers of ten since 10^{j-1}, 10^{j-2}, . . . , 10^2, 10^1, and 10^0 are not zero in modulo 2^j. Accordingly, only the last j digits need to be tested in order to determine if n is divisible by 2^j.

For example, let n = 15387248. Then, $2 \mid n$ (2 divides n without remainder) since $2 \mid 8$, $4 \mid n$ since $4 \mid 48$, $8 \mid n$ since $8 \mid 248$, $16 \mid n$ since $16 \mid 7248$, but $32 \nmid n$ since $32 \nmid 87248$.

The next test to be developed is for divisibility by powers of 5. Since $10 \equiv 0 \pmod 5$, then divisibility tests for 5 are analogous to those for powers of 2. Check the last j digits in order to determine the divisibility of n by 5^j.

For example, let n = 214365875. Then, $5 \mid n$ since $5 \mid 5$, $25 \mid n$ since $25 \mid 75$, $125 \mid n$ since $125 \mid 875$, but $625 \nmid n$ since $625 \nmid 5875$.

The next divisibility test to be developed is for powers of 3. Since $10 \equiv 1 \pmod 3$, then it follows that $10^k \equiv 1 \pmod 3$. Therefore,

$$
\begin{aligned}
n &= (a_k a_{k-1} \ldots a_1 a_0)_{10} \\
&= a_k 10^k + a_{k-1} 10^{k-1} + \ldots + a_2 10^2 + a_1 10^1 + a_0 10^0 \\
&\equiv a_k + a_{k-1} + \ldots + a_1 + a_0 \pmod 3.
\end{aligned}
$$

This indicates that the n is divisible by 3 if the sum of the digits of n is divisible by 3. The student should develop a similar divisibility test for 9 showing that 9 divides n if 9 divides the sum of the digits of n.

For example, let n = 347127. The sum of the digits of n is 3 + 4 + 7 + 1 + 2 + 7 = 24. Thus, $3 \mid n$ since $3 \mid 24$, but $9 \nmid n$ since $9 \nmid 24$.

The test for divisibility by 11 is found by observing that since $10 \equiv -1 \pmod{11}$, then

$$\begin{aligned} n &= (a_k a_{k-1} \ldots a_1 a_0)_{10} \\ &= a_k 10^k + a_{k-1}10^{k-1} + \ldots + a_2 10^2 + a_1 10^1 + a_0 10^0 \\ &\equiv a_k(-1)^k + a_{k-1}(-1)^{k-1} + \ldots + a_2 - a_1 + a_0 \pmod{11}. \end{aligned}$$

This shows that n is divisible by 11 if, the integer formed by alternately adding and subtracting the digits of n, is divisible by 11.

For example, let n = 24371853. Then alternately adding and subtracting the digits of n yields 2 - 4 + 3 - 7 + 1 - 8 + 5 - 3 = -11 which is divisible by 11.

Finally, a test for the divisibility of 7, 11, and 13 can be developed simultaneously. First, note that 7 x 11 x 13 = 1001 and that $10^3 = 1000 \equiv 1 \pmod{1001}$. Therefore,

$$\begin{aligned} n &= (a_k a_{k-1} \ldots a_1 a_0)_{10} \\ &= a_k 10^k + a_{k-1}10^{k-1} + \ldots + a_1 10^1 + a_0 10^0 \\ &= a_0 + 10a_1 + 100a_2 + 1000(a_3 + 10a_4 + 100a_5) + \\ &\quad 1000^2(a_6 + 10a_7 + 100a_8) + \ldots \\ &\equiv (a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + \\ &\quad (a_6 + 10a_7 + 100a_8) - \ldots \pmod{1001} \\ &\equiv (a_2a_1a_0)_{10} - (a_5a_4a_3)_{10} + (a_8a_7a_6)_{10} - \ldots \pmod{1001}. \end{aligned}$$

This indicates that n is congruent modulo 1001 if, the integer formed by alternately adding and subtracting successive blocks of three starting with the units digit, is divisible by 1001. Since 7, 11, and 13 are divisors of 1001, the check of their divisibility involves alternately adding and subtracting successive blocks of three in order to determine if the result is divisible by 7, 11, or 13.

For example, let n = 59358208. The alternating sum and difference of the blocks of three is equal to 208 - 358 + 59 = -91. Thus, $7 \mid n$ since $7 \mid -91$, $13 \mid n$ since $13 \mid -91$, but $11 \nmid n$ since $11 \nmid -91$.

Using the methods described above, the student should now be able to derive the divisibility rules for 6, 9, 10, and 12.

Conclusion

Congruences have been used since Gauss introduced them around the beginning of the nineteenth century. The anecdotes give the student practice in applying congruences. Congruences have many applications in everyday life as well as in number theory. The student should attempt to master the intricacies of congruence and try to discover their inter-relationships with the real world.

Diophantine Equations

Introduction

Diophantus was not the first mathematician to solve indeterminate problems, but he was the first to make an extensive study of the types of problems and equations that are associated with his name.[23]

Diophantus, the most famous Greek mathematician of his day, was known to have resided in Alexandria about 250 A.D. Many of the books and treatises which he had left for future generations of mathematicians have been lost. However, several books from his *Arithmetics* series, have been preserved.[24]

What little is known about the life of Diophantus comes from an epigram found in a collection called the *Greek Anthology* : "Diophantus passed one-sixth of his life in childhood, one-twelfth in youth, and one-seventh as a bachelor. Five years after his marriage was born a son who died four years before his father, at half his father's age." If x was the age at which Diophantus died, then the equation becomes

$$x = \frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4,$$

and Diophantus must have died at the age of 84.[25]

Diophantine equations may be divided into linear and nonlinear categories. In this course, general solutions will be found which will include all particular solutions of an initial set of conditions.

Linear Diophantine Equations[36]

The three theorems below, given without proof, will be helpful in solving linear Diophantine equations.

Theorem 5. For integers a and b, with b > 0, there exist unique integers q and r such that a = bq + r, where $0 \leq r < b$. In this equation, known as the Division Theorem, the integers q and r are called the quotient and remainder in the division of a by b.

Theorem 6. For non-zero integers a and b, there exist integers x and y such that ax + by = (a, b), the greatest common divisor of a and b. This is the famous Euclidean Algorithm.

Theorem 7. If a, b, and c are integers, then ax + by = c represents a linear Diophantine equation. Let d = (a, b). If $d \nmid c$, then ax + by = c has no integral solutions. If d | c, then ax + by = c has infinitely many integral solutions. If (x_0, y_0) is a particular solution of ax + by + c, and if n is an integer, then all solutions can be expressed as $(x_0 + [b/d]\, n, y_0 + [a/d]\, n)$.

A series of examples may be helpful in understanding how these Theorems can be used to solve various Diophantine equations along with their practical applications.

Problem: Solve 2x + 5y = 113.

Solution: By Theorem 5, the Diophantine has integral solutions since (2, 5) = 1 which divides 113.

$$\begin{array}{r} 2 \\ 2\overline{)5} \\ \underline{4} \\ 1 \end{array}$$

Using Theorem 3, the above division problem can be written as

$$1 = 5 - 2 \cdot 2. \qquad \textbf{(I)}$$

Multiplying both sides of (1) by 113 and shuffling the terms yields

$$2[-226] + 5[113] = 113. \qquad (2)$$

Adding +10n and -10n to (2) in a clever way yields

$$2[-226 + 5n] + 5[113 - 2n] = 113. \qquad (3)$$

Now, from (2), $x = -226$ and $y = 113$ is a particular solution of the Diophantine equation; and from (3), the general solution is $x = -226 + 5n$ and $y = 113 - 2n$. If only positive integral solutions are desired, then

$-226 + 5n > 0$	and	$113 - 2n > 0,$
$5n > 226$	and	$2n < 113,$
$n > 45.2$	and	$n < 56.5,$
$n \geq 46$	and	$n \leq 56.$

Therefore: $46 \leq n \leq 56$, and the solutions become (4, 21), (9, 19), (14, 17), (19, 15), (24, 13), (29, 11), (34, 9), (39, 7), (44, 5), (49, 3), (54, 1).

Problem: Solve $2x + 6y = 117$.

Solution: This Diophantine equation has no solution since $(2, 6) = 2$ does not divide 117.

Problem: Solve $666x + 1414y = 800$.

Solution: The Diophantine has integral solutions since $(666, 1414) = 2$ which divides 800.

$$\begin{array}{r} 2 \\ 666\overline{)1414} \\ \underline{1332} \\ 82 \end{array} \qquad \begin{array}{r} 8 \\ 82\overline{)666} \\ \underline{656} \\ 10 \end{array} \qquad \begin{array}{r} 8 \\ 10\overline{)82} \\ \underline{80} \\ 2 \end{array}$$

From Theorem 3,

$$\begin{aligned} 2 &= 82 - 8 \cdot 10, \\ 10 &= 666 - 8 \cdot 82, \\ 82 &= 1414 - 2 \cdot 666. \end{aligned}$$

Then,

$$\begin{aligned} 2 &= 82 - 8(666 - 8 \cdot 82) \\ &= 65 \cdot 82 - 8 \cdot 666 \\ &= 65(1414 - 2 \cdot 666) - 8 \cdot 666 \\ &= 65 \cdot 1414 - 138 \cdot 666. \end{aligned}$$

So,

$$666(-138) + 1414(65) = 2.$$
$$666(-55200) + 1414(26000) = 800 \text{ is the particular solution.}$$
$$666(-55200 + 1414\,n) + 1414(26000 - 666\,n) = 800 \text{ is the general solution.}$$

If only positive integral solutions are desired, then

$-55200 + 1414\,n > 0$	and	$26000 - 666n > 0$,
$1414\,n > 55200$	and	$666n < 26000$,
$n > 39.04$	and	$n < 39.04$,
$n \geq 40$	and	$n \leq 39$.

Therefore: x and y cannot both be positive. However, there are infinitely many solutions where x and y are opposite in sign.

Problem: A postal worker has only 14-cent and 21-cent stamps for sale. What combinations of these stamps will total $3.50?

Solution:

$$14x + 21y = 350;$$
$$(14, 21) = 7 \mid 350;$$
$$2x + 3y = 50;$$
$$2(10) + 3(10) = 50;$$
$$2(10 + 3n) + 3(10 - 2n) = 50.$$

$10 + 3n > 0$	and	$10 - 2n > 0$,
$3n > -10$	and	$2n < 10$,
$n > -3.33$	and	$n < 5$,
$n \geq -3$	and	$n \leq 5$.

Therefore: $-3 \leq n \leq 5$, and the solutions become (1, 16), (4, 14), (7, 12), (10, 10), (13, 8), (16, 6), (19, 4), (22, 2), and (25, 0).

Problem: Which combinations of pennies, dimes, and quarters have a value of 99¢?

Solution: The Diophantine that can be used to solve the given conditions is $P + 10D + 25Q = 99$. Let $W = P + 10D$. Then, $W + 25Q = 99$. In particular, $1(-1) + 25(4) = 99$. In general, $1(-1 + 25m) + 25(4 - m) = 99$. So, $W = -1 + 25m$ and $Q = 4 - m$. If $m = 0$, then $W = -1$; thus $P + 10D = -1$. By inspection, $1(9) + 10(-1) = -1$ and $1(-225m) + 10(25m) = +25m$. So $1(9 - 225m) + 10(25m - 1) = -1 + 25m$. In general, $1(9 - 225m + 10n) + 10(25m - 1 - n) = -1 + 25\,m$. Therefore, $P = 9 - 225m + 10n$, $D = 25m -$

1 - n, and Q = 4 - m. Requiring positive values,

$$9 - 225m + 10n \geq 0 \qquad 25m - 1 - n \geq 0 \qquad 4 - m \geq 0,$$
$$10n \geq 225m - 9 \qquad n \leq 25m - 1 \qquad m \leq 4$$
$$n \geq 22.5m - 0.9 \qquad\qquad 0 \leq m \leq 4.$$

So that, $22.5m - 0.9 \leq n \leq 25m - 1$ and $0 \leq m \leq 4$.

If m = 0, then n = -1 which gives negative values for P and D. If m = 1, then $22 \leq n \leq 24$. If m = 2, then $45 \leq n \leq 49$. If m = 3, then $67 \leq n \leq 74$. If m = 4, then $90 \leq n \leq 99$.

The solutions are given below in Table 4.

TABLE 4

Solutions to P + 10D + 25Q = 99

m	n	P	D	Q
1	22	4	2	3
1	23	14	1	3
1	24	24	0	3
2	45	9	4	2
2	46	19	3	2
2	47	29	2	2
2	48	39	1	2
2	49	49	0	2
3	67	4	7	1
3	68	14	6	1
3	69	24	5	1
3	70	34	4	1
3	71	44	3	1
3	72	54	2	1
3	73	64	1	1
3	74	74	0	1
4	90	9	9	0
4	91	19	8	0
4	92	29	7	0
4	93	39	6	0
4	94	49	5	0
4	95	59	4	0
4	96	69	3	0
4	97	79	2	0
4	98	89	1	0
4	99	99	0	0

Nonlinear Diophantine Equations

The generation of all possible Pythagorean triples can be achieved through the use of nonlinear Diophantine equations.[27] Consider the Pythagorean equation $x^2 + y^2 = z^2$, where the greatest common factor of x, y, and z is equal to one. Let $x + z = m$, and $z - x = n$, where m and n are integers. Then,

$$x = (m - n)/2,\ y^2 = mn, \text{ and } z = (m + n)/2.$$

In order to satisfy $y^2 = mn$, let $m = rp^2$, and $n = rq^2$, where r, p, and q are integers. Then,

$$x = r(p^2 - q^2)/2,\ y = pqr, \text{ and } z = r(p^2 + q^2)/2.$$

Since the greatest common factor of x, y, and z is equal to one, then r must be equal to one. Also, x, y, and z can be multiplied by two without affecting the validity of $x^2 + y^2 = z^2$. Thus,

$$x = p^2 - q^2,\ y = 2pq, \text{ and } z = p^2 + q^2,$$

and so the entire family of Pythagorean triples can be generated by assigning integral values to p and q.[28] Table 5 shows a partial listing of the Pythagorean triples that can be obtained noting that $p \neq q$ (otherwise x would equal zero), and that $p > q$ (otherwise, x would be negative).

TABLE 5

A Partial List of Pythagorean Triples

p	q	x	y	z
2	1	3	4	5
3	1	8	6	10
3	2	5	12	13
4	1	15	8	17
4	2	12	16	20
4	3	7	24	25
5	1	24	10	26
5	2	21	20	29
5	3	16	30	34
5	4	9	40	41
6	1	35	12	37
6	2	32	24	40
6	3	27	36	45

It is apparent that when p and q are not relatively prime, or when p and q differ by an even number, then the Pythagorean triple thus generated is a multiple of some previous one. This redundancy can be eliminated by requiring that x, y, and z have a greatest common factor of one. When $(x, y, z) = 1$, then $x^2 + y^2 = z^2$ is called a **primitive** Pythagorean triple.[29] To generate a primitive Pythagorean triple, some additional restrictions must be placed upon p and q. In order to verify the restrictions, some Theorems must be stated and proven.

Theorem 8: The square of an even number is congruent to zero modulo 4.

Proof: Let $n = 2k$ be any even number. Then, $n^2 = 4k^2 \equiv 0 \pmod 4$.

Theorem 9: The square of an odd number is congruent to one modulo 4.

Proof: Let $n = 2k + 1$ be any odd number. Then $n^2 = 4k^2 + 4k + 1 \equiv 0 + 0 + 1 \pmod 4 \equiv 1 \pmod 4$.

Corollary 1: The square of any number is congruent to 0 or 1 modulo 4.

Proof: Since numbers are either odd or even, this follows from Theorems 1 and 2.

Theorem 10: If x and y are both even, then the Pythagorean triple $x^2 + y^2 = z^2$ cannot be primitive.

Proof: Let $x = 2k$ and $y = 2j$. Then $x^2 = 4k^2$ and $y^2 = 4j^2$. Thus, $x^2 + y^2 = z^2 = 4k^2 + 4j^2 = 4(k^2 + j^2)$. And so, $z = 2\sqrt{k^2+j^2}$. But x, y, and z have a common factor of 2. Therefore, the Pythagorean triple cannot be primitive.

Theorem 11: If x and y are both odd, then the Pythagorean triple $x^2 + y^2 = z^2$ cannot exist.

Proof: Let $x = 2k + 1$ and $y = 2j + 1$. Then, $x^2 = 4k^2 + 4k + 1$ and $y^2 = 4j^2 + 4j + 1$. Thus, $x^2 + y^2 = z^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4(k^2 + k + j^2 + j) + 2 \equiv 0 + 2 \pmod 4 \equiv 2 \pmod 4$. But by Corollary 1, no number squared can be congruent to 2 modulo 4. Therefore, the Pythagorean triple cannot exist.

Corollary 2: If the Pythagorean triple $x^2 + y^2 = z^2$ is primitive, then x and y must have opposite parity.

Proof: If x and y are both even, then by Theorem 3, the Pythagorean triple is not primitive. If x and y are both odd, then Theorem 4 states that the Pythagorean triple cannot exist. Therefore, x and y must opposite parity.

The restrictions on p and q, the generators of the Pythagorean triple $x^2 + y^2 = z^2$, can now be verified.

Theorem 12: If $(p, q) \neq 1$, then $x^2 + y^2 = z^2$ is not primitive.

Proof: Assume that $(p, q) = d$ where d is an integer greater than one. Let $p = dk$ and $q = dj$. Then $p^2 = d^2k^2$ and $q^2 = d^2j^2$. Now, $x^2 = p^2 - q^2 = d^2k^2 - d^2j^2 = d^2(k^2 - j^2)$, $y^2 = 2pq = 2(dk)(dj) = d^2(2kj)$, and $z^2 = p^2 + q^2 = d^2k^2 + d^2j^2 = d^2(k^2 + j^2)$. And so, $x = d\sqrt{k^2 - j^2}$, $y = d\sqrt{2kj}$, and $z = d\sqrt{k^2 + j^2}$. Therefore, $(x, y, z) = d$, and thus $x^2 + y^2 = z^2$ is not primitive.

Corollary 3: If p and q are both even, then $x^2 + y^2 = z^2$ is not primitive.

Proof: If p and q are even, then by Theorem 5, $d = 2$, and therefore $x^2 + y^2 = z^2$ is not primitive.

Theorem 13: If p and q are both odd, then $x^2 + y^2 = z^2$ is not primitive.

Proof: Let $p = 2k + 1$ and $q = 2j + 1$. Then, $p^2 = 4k^2 + 4k + 1$ and $q^2 = 4j^2 + 4j + 1$. Now, $x^2 = p^2 - q^2 = 4k^2 + 4k + 1 - 4j^2 - 4j - 1 = 4(k^2 + k - j^2 - j) \equiv 0 \pmod 2$. Therefore, x is even. But, $y^2 = 2pq \equiv 0 \pmod 2$. So, y is even. But by Corollary 2, x and y must have opposite parity. Therefore, p and q cannot both be odd. So, in order for $x^2 + y^2 = z^2$ to be primitive, generators p and q must have a greatest common factor of one and opposite parity. Thus, a partial list of primitive Pythagorean triples can be generated.

TABLE 6

A Partial Listing of Primitive Pythagorean Triples

p	q	x	y	z
2	1	3	4	5
3	2	5	12	13
4	3	7	24	25
4	1	15	8	17
5	4	9	40	41
5	2	21	20	29
6	5	11	60	61
6	1	35	12	37
7	6	13	84	85
7	4	33	56	65
7	2	45	28	53
8	7	15	112	113
8	5	39	80	89
8	3	55	48	73
8	1	63	16	65
9	8	17	144	145
9	4	65	72	97
9	2	77	36	85

The preceding problem dealt with nonlinear Diophantine equations of the form $x^2 + y^2 = z^2$. Another problem to be considered is to find pairs of integers x and y such that $x^2 + y^2 = z$. In other words, what integers can be expressed as the sum of two squares? For example, the integer 29 can be written as $2^2 + 5^2$, but the integer 19 cannot be written as the sum of two squares. A look at the first twenty positive integers reveals:

$1 = 0^2 + 1^2$,
$2 = 1^2 + 1^2$,
3 is not the sum of two squares,
$4 = 2^2 + 0^2$,
$5 = 1^2 + 2^2$,
6 is not the sum of two squares,
7 is not the sum of two squares,
$8 = 2^2 + 2^2$,
$9 = 3^2 + 0^2$,
$10 = 3^2 + 1^2$,
11 is not the sum of two squares,
12 is not the sum of two squares,
$13 = 3^2 + 2^2$,
14 is not the sum of two squares,
15 is not the sum of two squares,
$16 = 4^2 + 0^2$,
$17 = 4^2 + 1^2$,
$18 = 3^2 + 3^2$,
19 is not the sum of two squares,
$20 = 2^2 + 4^2$.

It is not difficult to recognize numbers which *cannot* be written as the sum of two squares. Recall that the square of an even integer is congruent to 0 modulo 4, while the square of an odd integer is congruent to 1 modulo 4. Therefore, the sum of the squares of two even integers must be congruent to $0 + 0 \equiv 0 \pmod 4$; the sum of the squares of two integers of opposite parity is congruent to $0 + 1 \equiv 1 \pmod 4$; or the sum of the squares of two odd integers is congruent to $1 + 1 \equiv 2 \pmod 4$. So, an integer of the form $4k + 3$, which is $3 \pmod 4$, cannot be expressed as the sum of two squares. This includes numbers such as 3, 7, 11, 15, or 19.

However, even integers, or integers of the form $4k + 1$, may or may not be able to be expressed as the sum of two squares. To determine if z can be expressed as the sum of two squares, it is necessary to factor z into its prime factors. Note that these factors must include powers of the integer 2, or powers of integers having the form $4k + 1$ or $4k +3$. It is important to know which of these primes can be expressed as the sum of two squares.[30]

It is easy to see that 2 can be expressed as the sum of two squares since $2 = 1^2 + 1^2$. Next to be considered are primes of the form $4k + 1$. Fermat stated, and Euler proved a century later, that every prime number of the form $4k + 1$ can be expressed as the sum of two primes. Finally, as proven earlier, primes of the form $4k + 3$ cannot be expressed as the sum of two squares.

These three results can be applied to a composite number with the help of Theorem 14.

Theorem 14: If integers m and n can each be expressed as the sum of two squares, then their product, m·n, can be expressed as the sum of two squares.

Proof: Let $m = a^2 + b^2$ and $n = c^2 + d^2$. Then, $m \cdot n = (a^2 + b^2)(c^2 + d^2) = a^2 c^2 + b^2 d^2 + a^2 d^2 + b^2 c^2 = a^2 c^2 + 2abcd + b^2 d^2 + a^2 d^2 - 2abcd + b^2 c^2 = (ac + bd)^2 + (ad - bc)^2$. Therefore, m·n can be expressed as the sum of two squares.

Theorem 6 accounts for combinations of primes which *can* be expressed as the sum of two primes. Theorems 7 and 8 will deal with combinations of primes which *cannot* be expressed as the sum of two primes.

Theorem 15: If an integer contains a prime factor of the form 4k + 3, then in order for the integer to be written as the sum of two squares, this prime factor must occur an even number of times.

Proof: $(4k + 3)^{2n} = [(4k + 3)^2]^n = (16k^2 + 24k + 9)^n \equiv [1(\text{mod } 4)]^n$. Since 1 (mod 4) is a perfect square, it can be expressed as the sum of two squares, itself and zero. Therefore, $[1(\text{mod } 4)]^n$ can be expressed as the sum of two squares since Theorem 6 can be applied n times.

Theorem 16: If an integer contains a prime factor of the form 4k + 3 an odd number of times, then the integer cannot be expressed as the sum of two squares.

Proof: $(4k + 3)^{2n+1} = (4k + 3)^{2n}(4k + 3)^1 \equiv 1 \cdot 3 \ (\text{mod } 4) \equiv 3 \ (\text{mod } 4)$.

Therefore, $(4k + 3)^{2n+1}$ cannot be expressed as the sum of two squares.

The preceding results may be summarized as follows: An integer can be expressed as the sum of two squares if its prime factors do *not* contain an odd number of any primes of the form 4k + 3.

Problem: Show that 130 can be written as the sum of two squares.

Solution: 130 = 2 x 5 x 13. All prime factors are either 2, or powers of the form 4k + 1. In addition, since $130 \equiv 2 \ (\text{mod } 4)$, both squares must be odd.

Therefore: $130 = 11^2 + 3^2$.

Problem: Show that 72 can be written as the sum of two squares.

Solution: $72 = 2^3 \times 3^2$. All prime factors are either powers of 2, or powers of the form 4k + 3 to an even power. In addition, since $72 \equiv 0 \ (\text{mod } 4)$, both squares must be even.

Therefore: $72 = 6^2 + 6^2$.

Problem: Show that 84 can be written as the sum of two squares.

Solution: $84 = 2^2 \times 3 \times 7$. All prime factors are either 2, or odd powers of the form $4k + 3$.

Therefore: 84 cannot be expressed as the sum of two squares.

The student should consider the sum of the squares of the numbers from one to one-hundred.

Finally, no introduction to Diophantine equations would be complete without mention of the Pell equation. The Pell equation is a particular type of Diophantine equation named for the mathematician who first focused on it.[31]

The simplest form of the Pell equation is $x^2 - 2y^2 = 1$. This equation has an infinite number of solutions. Some of the successive pairs of x and y values are:

x	1	3	17	99	577	etc.
y	0	2	12	70	408	etc.

Notice the following relationship between the respective values of x and y beginning with the third set of solutions:

$17 = 6 \cdot 3 - 1$, $99 = 6 \cdot 17 - 3$, $577 = 6 \cdot 99 - 17$, etc.
$12 = 6 \cdot 2 - 0$, $70 = 6 \cdot 12 - 2$, $408 = 6 \cdot 70 - 12$, etc.

Note that $x = 3$ is the value of x in the first non-zero solution. Note also that $3 \times 2 = 6$.

Next, look at the equation $x^2 - 3y^2 = 1$. Some of the successive values of x and y pairs are:

x	1	2	7	26	97	etc.
y	0	1	4	15	56	etc.

Notice the following relationship between the respective values of x and y beginning with the third set of solutions:

$7 = 4 \cdot 2 - 1$, $26 = 4 \cdot 7 - 2$, $97 = 4 \cdot 26 - 7$, etc.
$4 = 4 \cdot 1 - 0$, $15 = 4 \cdot 4 - 1$, $56 = 4 \cdot 15 - 4$, etc.

Note that $x = 2$ is the value of x in the first non-zero solution. Note also that $2 \times 2 = 4$.

At first glance, the next equation would appear to be $x^2 - 4y^2 = 1$. However, since 4 is a perfect square, this equation could be written as $(x + 2y)(x - 2y) = 1$. This leads to a system of simultaneous equations:

$$x + 2y = 1$$
$$x - 2y = 1$$

which has only the solution (1,0), and hence no non-zero solutions. Similarly, equations such as $x^2 - 9y^2 = 1$, $x^2 - 16y^2 = 1$, ..., $x^2 - Ay^2 = 1$, where A is a perfect square, will not be considered here.

So, the next equation to be considered is $x^2 - 5y^2 = 1$, which has the following successive pairs of solutions:

x	1	9	161	2489	etc.
y	0	4	72	1292	etc.

Notice the following relationship between the respective values of x and y beginning with the third set of solutions:

$161 = 18 \cdot 9 - 1$, $\quad 2489 = 18 \cdot 161 - 9$, $\quad$ etc.
$72 = 18 \cdot 4 - 0$, $\quad 1292 = 18 \cdot 72 - 4$, $\quad$ etc.

Note that $x = 9$ is the value of x in the first non-zero solution. Note also that $9 \times 2 = 18$.

In general, the Pell equation $x^2 - Ay^2 = 1$, where A is not a perfect square, yields successive pairs of integral solutions, after the first non-zero pair, of the form:

$$x_k = 2ax_{k-1} - x_{k-2}$$
$$y_k = 2ay_{k-1} - y_{k-2}$$

where $x = a$ is the value of x in the first non-zero solution.

This general solution enables the student to find successive pairs of solutions once the non-zero solution has been found. Most textbooks give a far more complex and impractical method of finding these solutions. It is usually quicker, and far more practical, to find the first non-zero solution by trial and error, and then to follow the method outlined here. It is a sound method, and is almost always more practical.

The most general form of the Pell equation is $x^2 - Ay^2 = B$ where A is a positive integer other than a perfect square, and B is a positive or negative integer. There will not always be integral solutions for all values of A and B. In fact, for B = -1, there will only be integral values for A = 2, 5, 10, 13, 17, etc. And for A = 3, there will only be integral values for B = 1, -2, -3, 4, 6, -8, etc. The technique for solving the most general form of the Pell equation is outlined below.

Assume that $x^2 - Ay^2 = B$ has integral solutions over certain values of A and B. Let (a, b) be the first non-zero solution of $x^2 - Ay^2 = B$, and let (c, d) be any integral solution of $x^2 - Ay^2 = 1$. Then, $x^2 - Ay^2 = (a^2 - Ab^2)(c^2 - Ad^2) = a^2c^2 + A^2b^2d^2 - Aa^2d^2 - Ab^2c^2 =$ $a^2c^2 \pm 2Aabcd + A^2b^2d^2 - (Aa^2d^2 \pm 2Aabcd + Ab^2c^2) = (ac \pm Abd)^2 - A(ad \pm bc)^2$.

Therefore, $x = ac \pm Abd$ and $y = ad \pm bc$. By substituting, for c and d, any pair of values which satisfies $c^2 - Ad^2 = 1$, solutions to the original equation $x^2 - Ay^2 = B$ can be obtained.

For example, in the equation $x^2 - 3y^2 = -11$, the smallest integral non-zero solution, by trial and error, is (1, 2). Then, $x = \pm(c \pm 6d)$ and $y = \pm(d \pm 2c)$ where c and d are integers which satisfy $c^2 - 3d^2 = 1$.

Some values of x and y generated by successive values of c and d are:

c	1	2	2	7	7	26	26	etc.
d	0	1	1	4	4	15	15	etc.
x	1	4	8	17	31	64	116	etc.
y	2	3	5	10	18	37	67	etc.

It is possible that not all integral solutions can be found using the procedure outlined above. In this case, the procedure must be slightly modified. For example, try to find all integral solutions of $x^2 - 2y^2 = 119$ for values of x less than 200.

By trial and error, the smallest non-zero solution is (11, 1). Then, $x = \pm(11c \pm 2d)$ and $y = \pm(m \pm 11d)$ where $c^2 - 2d^2 = 1$. The solutions are:

x	11	29	37	163
y	1	19	25	115

But there may be other solutions. If there are other solutions, they would be found between the smallest and next smallest solutions found in the first tabulation. Therefore, the procedure is to check for y values between 1 and 19. This can be done quickly with the result that x = 13 and y = 5. This generates a new series of solutions where $x = \pm(13c \pm 10d)$ and $y = \pm(5c \pm 13d)$ where $c^2 - 2d^2 = 1$.

The additional solutions become:

x	13	19	59	101
y	5	11	41	71

The search for a third family need not be initiated as a result of a useful rule which states that if B is the product of n prime numbers, then there will be, at most, 2^{n-1} families of integral solutions. In this example, since 119 = 7 x 17, then n = 2 and, therefore, there are 2 such families which, in this case, have already been identified.

Conclusion

Thus ends the discussion of Diophantine equations. Such equations arise in both practical and recreational problems. Only a small part of the entire field of Diophantine equations has been discussed here. It is hoped that the student will have been given some insight into the vast field of Diophantine equations.

Additional Topics

The Fibonacci Sequence

Number sequences have long provided mathematicians with thought-provoking problems and interesting applications to the real world. One particular sequence, the Fibonacci sequence, is especially interesting and powerful in its mathematical applications.[32] The Fibonacci sequence is produced by starting with 1 and adding the previous two numbers in the sequence in order to produce the next number in the sequence: 1, 1, 2, 3, 5, 8, 13, 21, ... The Fibonacci sequence can be represented by a recursive formula. If F_n represents the nth Fibonacci number, then $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

The Fibonacci sequence was given as a solution by Leonardo of Pisa (Fibonacci) to a famous problem which shall be renamed the "amoeba problem". A baby amoeba, called "a", takes a day to mature into an adult, called "A".[33] As an adult, it asexually reproduces to produce a baby amoeba, and, at the same time it continues to live. After a day, the baby becomes an adult while the original adult reproduces again. This continuing process is illustrated below.

```
            a
            |
            A
           / \
          A   a
         / |   \
        A  a    A
       /|  |    | \
      A a  A    A  a
     / | | |\  |\   \
    A  a A A a a A   A
   / | / / | /\ | | |\ |\
  A a A A a A a A A a A a A
```

FIGURE 1

Family "Tree" of a Single Baby Amoeba

The student can continue the diagram through several more generations. In doing so, notice that the number of amoebae present on successive days form a Fibonacci sequence. For example, there is one amoeba present on day one, followed by one again on day two. Then there are two on day three, followed by, three, five, eight, thirteen, and so on. The sequence could be continued indefinitely because the Fibonacci sequence is an infinite sequence.

Fibonacci numbers appear in many probability problems related to the binomial theorem. The student should recall Pascal's triangle which is used in order to simplify binomial expansions. Here, a different application of Pascal's triangle can be used. Notice how the Fibonacci sequence can be derived by summing the diagonals in Pascal's triangle as illustrated in figure 2 on the following page.

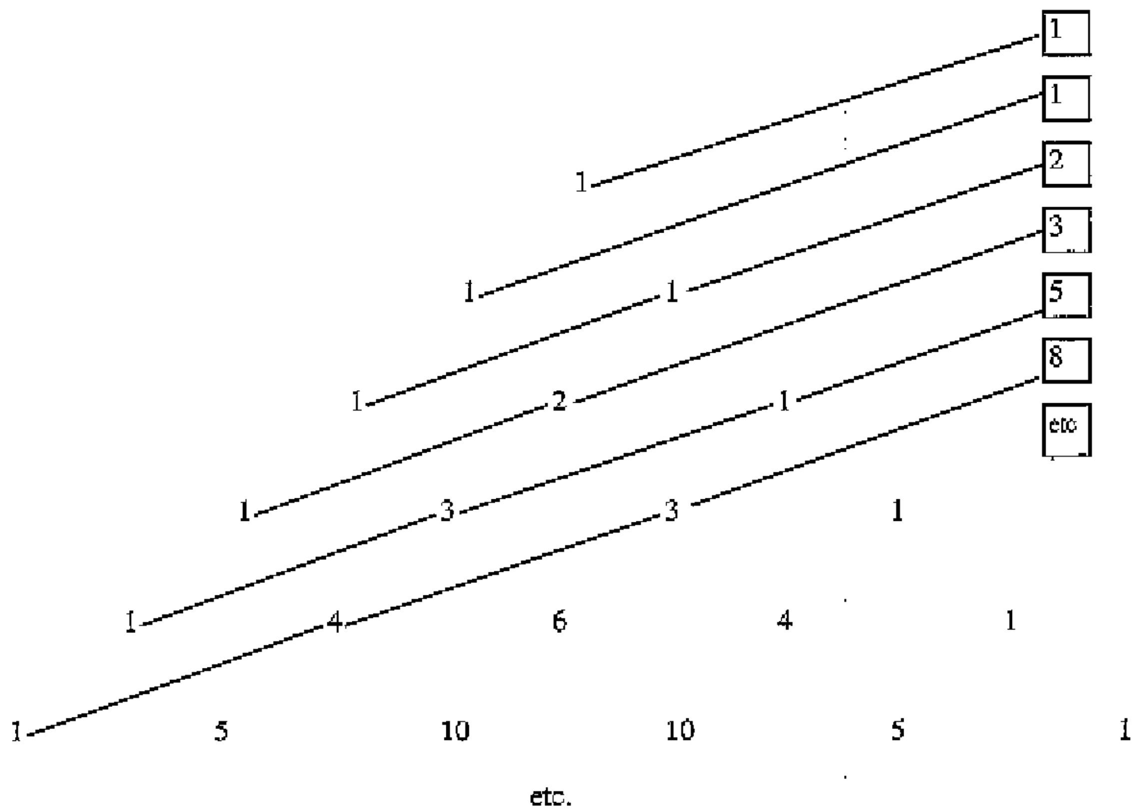

FIGURE 2

The Fibonacci Sequence from the Diagonals of Pascal's Triangle

Problem: If a coin is flipped N times, how many unique sequences of heads and tails are possible provided that the coin cannot come up heads twice in a row?

Solution:

TABLE 7

Sequences of Heads and Tails without Consecutive Heads

Number of Flips	Sequences	Total
1	{H}, {T}	2
2	{HT}, {TH}, {TT}	3
3	{HTH}, {HTT}, {THT}, {TTH}, {TTT}	5
4	{HTHT}, {HTTH}, {HTTT}, {THTH}, {THTT}, {TTHT}, {TTTH}, {TTTT}	8

Note that the total number of outcomes is always a Fibonacci number.

In nature, Fibonacci numbers have been found to be associated with natural spirals in objects such as pine cones, daisy blossoms, and pineapples. In the daisy blossom, for example, there are two sets of spirals. The clockwise set contains 21 spirals, while the counterclockwise set contains 34 spirals. Both are Fibonacci numbers.

In addition, Fibonacci numbers are found in patterns of the leaves and branches of many different species of trees. This phenomenon is called phyllotaxis where the arrangement of leaves around a stem can be expressed as a fraction:

$$\frac{\text{number of complete turns}}{\text{number of leaves per cycle}}$$

For example, the cherry and oak have a phyllotaxis of 2/5, while the beech has a phyllotaxis of 1/3. Other examples are pear, 3/8, and the willow, 5/13.[34]

There are many excellent references dealing with Fibonacci sequences. Some of these include, Fun with Mathematics by Jerome Meyer, Mathematical Diversions by J. A. H. Hunter and Joseph Madachy, and Mathematical Ideas by Charles D. Miller, Vern E. Heeren, and E. John Hornsby, Jr. The student should consult these and many other books for more applications of the Fibonacci sequence, which include their relationship with the golden ratio, examined in the following section.

The Golden Section

The Paprus of Ahmes, inscribed hundreds of years before the rise of ancient Greek culture, contains a detailed account of the building of the Great Pyramid of Gizeh. The account refers to a "sacred ratio" of the slant edge length to the distance from the base edge to the ground center which was equal to 1.618. This ratio is the Golden Section of the ancient Greeks.[35]

If a line segment is partitioned so that the larger part is the mean proportion between the shorter part and the entire segment, then the ratio of the larger part to the smaller part is called the golden section. Referring to Figure 3 on the following page, if the larger part of the line segment is called "x" and the lesser part of the line segment is called "y", then

$$\frac{x+y}{x} = \frac{x}{y}.$$

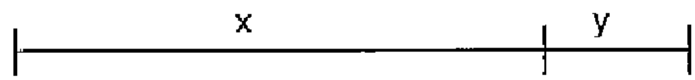

FIGURE 3

A Line Segment Partitioned into a Greater and Lesser Part

Thus, $x^2 - xy - y^2 = 0$, and therefore $\frac{x}{y} = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ is the golden section.

Another approach that leads to the golden section can be seen from the regular pentagon in Figure 4 below.[36]

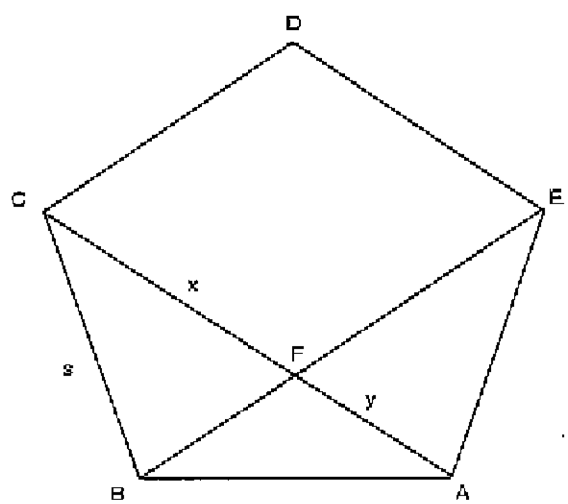

FIGURE 4

A Regular Pentagon

In regular pentagon ABCDE, of side s, diagonals AC and BE intersect at F, with CF = x and AF = y. Since Δ ABC is isosceles with $\angle$ ABC = 108°, then $\angle$ BCA = 36°. Also, Δ ABF is isosceles, and so, $\angle$ CBF = 108° - 36° = 72° and $\angle$ CFB = 180° - 72° - 36° = 72°. Hence, CF = CB, and therefore, x = s. If an altitude is drawn from B to AFC, then it can be easily shown that $x + y = 2s \cos 36$. Similarly, drawing an altitude from F to AB, yields $y = \frac{s}{2\cos 36}$. Then, $x = \frac{s(4\cos^2 36 - 1)}{2\cos 36}$. But, since $x = s$, then

$4\cos^2 36 - 1 = 2\cos 36$ or $4\cos^2 36 - 2\cos 36 - 1 = 0$ with positive solution:

$$\cos 36 = \frac{1}{4}(1 + \sqrt{5}).$$

Then, $x = s$ and $y = \frac{2s}{(1+\sqrt{5})}$, and so, $\frac{x}{y} = \frac{1}{2}(1+\sqrt{5})$, the golden section.

Problem: Prove, that in Figure 5 below, AC : AB = AC : AN = AN : NC = AM : MN = OD : DF = the golden section.

Solution: This is to be left as an exercise for the student.

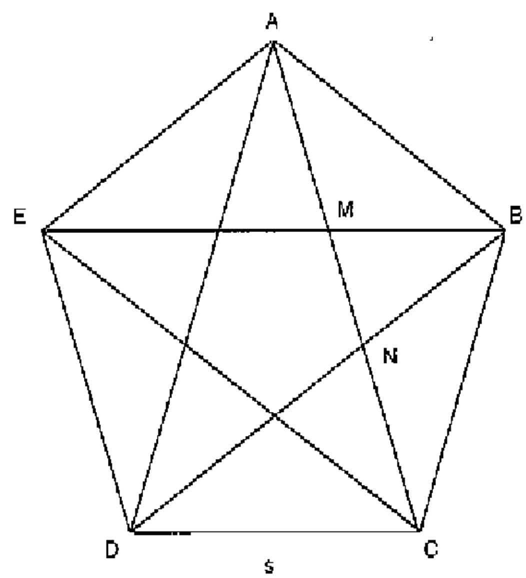

FIGURE 5

A Star within a Pentagon

At this point, two curious aspects of the golden section may be noted. First, it is easy to see with a calculator that the golden ratio can be transformed into its reciprocal merely by subtracting 1 from it, yielding 0.618 . . . . Secondly, if the height of one's body is divided by the height of one's navel, a number very close to 1.618 is obtained.[37]

A final method presented here to discover the golden section comes from dividing the nth term of a Fibonacci sequence by the (n - 1)th term and observing the result. As n increases to infinity, the ratio obtained converges to the golden section. A look at the first twelve such ratios are shown in Table 8 below.[38]

TABLE 8

The First Twelve Ratios for Successive Fibonacci Numbers to Four Decimal Places

1/1 = 1.0000	21/13 = 1.6154
2/1 = 2.0000	34/21 = 1.6190
3/2 = 1.5000	55/34 = 1.6176
5/3 = 1.6667	89/55 = 1.6182
8/5 = 1.6000	144/89 = 1.6180
13/8 = 1.6250	233/144 = 1.6181

Many Renaissance mathematicians became intrigued with the golden section. H. S. M. Coxeter quotes Kepler as follows: "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."[39]

Further study of the golden section should be pursued by the student. Some of the possible areas of investigation include inscribed decagons, golden rectangles (rectangles with sides in golden ratio), icosahedrons, dodecahedrons, and logarithmic spirals. Some references include: Der goldene Schnitt by Adolf Zeising; Nature's Harmonic Unity by Samuel Colman; The Curves of Life by Sir Theodore Cook; Mathematical Puzzles and Diversions by Martin Gardner; Mathematical Diversions by J. A. H. Hunter and Joseph S. Madachy; and Mathematical Ideas by Charles D. Miller, Vern E. Heeren, and E. John Hornsby, Jr.

The Imaginary Number *i*

An imaginary number is a precise mathematical idea. It forced itself into algebra in much the same way as did the negative numbers. Raphael Bombelli of Bologna saw that equations of the form $x^2 + a = 0$, where a is any positive number, could not be solved without the aid of imaginaries.[40]

Thus, $i = \sqrt{-1}$, and therefore, $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $i^5 = i$, . . . and the process keepsrecycling. A number of the form a + bi is called a complex number and is a mixture of a real and an imaginary number.

A most fascinating discovery was that every number has n nth roots.[41] The number 64, for example, has 2 square roots, 3 cube roots, 4 fourth roots, and so on. Many people may know that the square roots of 64 are +8 and -8. But perhaps less people know that the cube roots of 64 are 4, $-2 + 2i\sqrt{3}$, and $-2 - 2i\sqrt{3}$. Thus, the concept of complex numbers is introduced. The general form for finding the nth roots of x is given by :

$$x^{1/n} = x^{1/n} (\cos 2k\pi/n + i \sin 2k\pi/n) \text{ with } k = 0, 1, 2, \ldots, n - 1.$$

The student should attempt to find the fourth and fifth roots of 32 using this formula.

Imaginary numbers occur in many other applications in mathematics and physics including electronics. The student should pursue the properties of *i* in various mathematical and physics references such as the ones listed in the section on π.

The Exponential Function

In 1614, John Napier issued his *Mirifici Logarithmorum Canonis Descriptio,* the first treatise on logarithms.[42] His invention may have been as important to mathematicians as Arabic numerals. If logarithms had not been discovered, mathematics, astronomy, and physics would have been put back a century or more.[43] Since *e* and logarithms are closely related, a close look at logarithms should reveal something about the nature of *e*.

The two progressions:

Arithmetic--	0	1	2	3	4	5	6	7	8	9 . . .
Geometric--	1	2	4	8	16	32	64	128	256	512 . . .

share the following relationship. If the terms of the arithmetic progression are regarded as exponents of 2, then the corresponding terms of the geometric progression represent the powers of 2. Thus, in base 2, each term in the arithmetic progression is the logarithm of the corresponding term in the geometric progression.[44]

Extensive tables of logarithms have been constructed in base 10 and in base *e* , the Napierian or natural base. Like π, the number *e* is transcendental. The most familiar infinite series of *e* is given by:

$$e = 1 + 1/1! + 1/2! + 1/3! + 1/4 + \ldots.$$

Thus, the value of *e* may be ascertained to as much accuracy as desired. To the tenth decimal place, *e* = 2.7182818285.

The student should try to prove that as n goes to infinity, then $(1 + 1/n)^n$ will produce the infinite series of *e*.

The constant *e* also plays an important role in the derivation of the tables of the sine and cosine functions with the help of *i*. Since $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$, then substituting iθ for x gives

$$\begin{aligned} e^{i\theta} &= 1 + i\,\theta + (i\,\theta)^2/2! + (i\,\theta)^3/3! + \ldots \\ &= 1 + i\,\theta - \theta^2/2! - i\,\theta^3/3! + \ldots \end{aligned}$$

since $i^2 = -1$ and $i^3 = -i$.

Noting that every other term contains *i* , and remembering that Leonhard Euler, a Swiss mathematician in the eighteenth century[45], proved that $e^{i\theta} = \cos\theta + i\sin\theta$, then:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \text{ and } \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

where θ is measured in radians.

Other applications of *e* can be found in various disciplines of science and mathematics such as: physics, chemistry, biology, calculus, number theory, and economics. Some applications include: RC series circuits, RL series and parallel circuits, half-life, reaction rates, bacterial growth and decay, cryptology, compound interest, and annuities. The student should do some research in some of these applications using the references mentioned following the discussion on π.

Pi

Archimedes, who lived in the second century B.C., proved that the value of π was less than 22/7 and greater than 223/71 by using a regular polygon of 96 sides. Ptolemy, in 150 A.D., used the value of 3.1416 for π. By the middle of the sixteenth century, the fraction 355/113 was discovered, giving the value of π to six decimal places. By the early seventeenth century, van Ceulen, a German mathematician, calculated π to more than 20 decimal places and got 3.141592653589793238464. Since the invention of calculus and the discovery of infinite series, the value of π can be calculated to any number of decimal places desired.[46] An exact expression for $\pi/4$ is given by the infinite series:

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots.$$

The value of π even occurs in the laws of chance. An experiment that the student may wish to try is the famous Buffon experiment. To perform this experiment, a needle and a horizontal surface ruled by a grid of parallel equidistant lines are needed. The distance h between the lines and the length l of the needle must satisfy the relationship $l \leq h$. Toss the needle so that it lands at random angles with respect to the parallel lines. After each toss, note whether the needle intersects with any of the parallel lines. Let m be the number of times the needle makes an intersection in n throws. Then π can be approximated by using the equation $\pi = \frac{2l}{h} \times \frac{n}{m}$. As n increases, the value of π is more closely approximated.[47]

One of the most famous problems in mathematical history is the "squaring of the circle". The problem is to construct a square equal to the area of a given circle using only a straightedge and a compass. The Greeks, and later mathematicians, sought such a construction but always failed. The German mathematician Lindemann, in 1882, published a proof that π was a transcendental number, and thus confirming that the circle can never be squared.[48]

These and numerous other allusions to the irrational number π should be investigated by the student. References are plentiful and include: Mathematics and the Imagination by Edward Kasner and James Newman; Mathematics for the Million by Lancelot Hogben; A New Look at Arithmetic by Irving Adler; and Fun with Mathematics by Jerome S. Meyer.

Perfect Numbers

Six is the first "perfect" number. The Greeks called it perfect because it is the sum of its proper divisors. The next four perfect numbers are 28, 496, 8128, and 33,550,336. The student is asked to verify that these five numbers are indeed perfect. It took more than two thousand years for mathematicians to find the next seven. Then, in 1952, a University of California professor discovered the first new perfect number in seventy-five years and, in the next few months, he discovered four more for a total of seventeen.[49]

It may be noteworthy that the Pythagoreans hailed ten as "perfect", but not in the way that six is. It had the special charm that it is the sum of one (the point), two (the line), three (the plane), and four (the solid).[50]

There are still many unanswered questions about perfect numbers. It is not known if there are infinitely perfect numbers; all known perfect numbers are even; it is not known if any odd perfect numbers exist; and an even number is perfect if and only if it is of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is called a Mersenne prime assuming that n is prime.[51]

The Mersenne primes have always received special attention because they are closely related to the perfect numbers.[52] The student should verify that the first seven Mersenne primes are: $M_2 = 3$, $M_3 = 7$, $M_5 = 31$, $M_7 = 127$, $M_{13} = 8{,}191$, $M_{17} = 131{,}071$, and $M_{19} = 524{,}287$. So, the problem of finding a new even perfect number is essentially the same as finding a new Mersenne prime. Thus, since M_3 is a Mersenne prime, then $2^2 \times M_3 = 28$ is a perfect number. Similarly, since M_5 is a Mersenne prime, then $2^4 \times M_5 = 496$ is a perfect number.

There are more interesting facts about perfect numbers: all known perfect numbers, except six, have digital roots of 1; every known perfect number, except six, is the sum of consecutive odd cubes; and all perfect numbers are the sum of successive powers of 2.[53]

If a number is not perfect, then it is either deficient or abundant. A counting number is deficient if it is greater than the sum of its proper divisors. It is abundant if it is less than the sum of its proper divisors.[54]

There is a class of multiperfect numbers in which the sum of the divisors of the number is a multiple of the number. For example, the sum of the divisors of 120 is 1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 20 + 24 + 30 + 40 + 60 = 240.[55]

Not quite as old as perfect numbers, but quite old are the amicable numbers.[56] The counting numbers a and b are amicable, or friendly, if the sum of the proper divisors of a is b, and the sum of the proper divisors of b is a. The smallest pair of amicable numbers, 220 and 284, was known to the Pythagoreans, but it was not until over one thousand years later that the next pair, 18,416 and 17,296, was discovered.[57]

The student may find more information on perfect numbers in the following references: Mathematical Diversions by J. A. H. Hunter and Joseph S. Madachy; From Zero to Infinity by Constance Reid; Mathematical Ideas by Charles D. Miller, Vern E. Heeren, and E. John Hornsby, Jr.; Number--The Language of Science by Tobias Dantzig; A New Look at Arithmetic by Irving Adler; and Elementary Number Theory by Kenneth H. Rosen.

The Shapes of Numbers

To the Pythagoreans, the important secrets of nature could all be expressed by simple relationships among the whole numbers. Numbers were very real to the Pythagoreans, and they had very distinctive shapes. The four most important shapes were triangular, square, oblong, and gnomons.[58]

Triangular numbers are those numbers that can be fitted into a triangle. The triangular numbers are one, three, six, ten, . . . , and are written as follows:

```
                              *
                    *        * *
          *        * *      * * *
*        * *      * * *    * * * *
```

The nth triangular number (T_n) is equal to the sum of the first n integers. For example, $T_2 = 1 + 2 = 3$; $T_3 = 1 + 2 + 3 = 6$; and in general, $T_n = n\,(n+1)/2$. Thus, the 6th triangular number would be: $T_6 = (6)(7)/2 = 21$.

Square numbers are those numbers that can be fitted into a square. The square numbers are one, four, nine, sixteen, . . . , and are written as follows:

```
                              * * * *
                    * * *     * * * *
          * *       * * *     * * * *
*         * *       * * *     * * * *
```

The nth square number (S_n) is equal to the sum of the first n odd integers. For example, $S_2 = 1 + 3 = 4$; $S_3 = 1 + 3 + 5 = 9$; and in general, $F_n = n^2$. Thus, the 6th square number would be: $S_6 = 6^2 = 36$.

Oblong numbers are those numbers that can be fitted into an n by n + 1 rectangle. The oblong numbers are two, six, twelve, twenty, . . . , and are written as follows:

```
                                    * * * * *
                      * * * *       * * * * *
          * * *       * * * *       * * * * *
* *       * * *       * * * *       * * * * *
```

The nth oblong number (O_n) is equal to the sum of the first n even integers. For example, $O_2 = 2 + 4 = 6$; $O_3 = 2 + 4 + 6 = 12$; and in general, $O_n = n\,(n + 1)$. Thus, the 6th oblong number would be: $O_6 = (6)(7) = 42$.

The gnomons are all of the odd numbers, and they are written in the shape of a gnomon or carpenter's square:

```
                                   *
                          *        *
                  *       *        *
          *       *       *        *
*       * *     * * *   * * * *  * * * * *
```

Various kinds of numbers can be combined. For example: a square number is always the sum of successive gnomons; any two successive triangular numbers also make a square number; and any two equal triangular numbers make an oblong number. The student should verify these facts with diagrams.

There are other families of numbers to be investigated. Expanding in two dimensions, there are pentagonal, hexagonal, . . . numbers. Expanding in three dimensions, there are multi-layered tetrahedral, pyramidal, . . . numbers. An excellent book on the subject is <u>Mathematics for the Million</u> by Lancelot Hogben. The student can develop both free-form and recursive formulas for all of these numbers with shapes.

<u>Cryptology</u>

<u>Introduction to Cryptology</u>

Ciphers, or secret messages, have been sent among people since antiquity. While the need for secret communication has traditionally occurred in both diplomatic and military affairs, electronic advances have prompted secrecy in areas such as banking and sports.[59]

The following terms should be defined since they will be used throughtout this discussion on cryptology. *Cryptology* is the study of secrecy systems. *Plaintext* is the message that will be altered and will be denoted by "P". *Ciphertext* is the altered text and will be denoted by "C". The *key* is the transformation to be used. *Enciphering* is the process of transforming P into C. *Deciphering* is the process of transforming C into P.

The following ciphers represent only a few of the many ciphers in existence. In fact, the student may want to design a cipher that can be used among friends. The ciphers will appear in order of increasing complexity.

Types of Ciphers

The first type of cipher to be studied is the simple shift cipher.[50] The enciphering key is mathematically expressed as $C \equiv P + k \pmod{25}$. To encipher, group the letters of the plaintext into groups of five. The purpose of this is to prevent a potential codebreaker from recognizing familar word patterns. Then, write the number equivalents of the plaintext. Next, use the shift transformation to find the number equivalents of the ciphertext. Finally, translate the numbers into the ciphertext.

Specifically, if $k = 6$, then the following relation is shown between the plaintext and the ciphertext.

	00	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
plaintext:	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	00	01	02	03	04	05
ciphertext:	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F

For example, encipher the following message using the transformation $C \equiv P + 6 \pmod{25}$.

THIS IS A SIMPLE SHIFT CIPHER.

Breaking the plaintext into groups of five letters gives:

THISI SASIM PLESH IFTCI PHER.

Converting the plaintext into number equivalents yields

19 07 08 18 08 18 00 18 08 12 15 11 04 18 07
08 05 19 02 08 15 07 04 17.

Using the transformation $C \equiv P + 6 \pmod{25}$ yields

25 13 14 24 14 24 06 24 14 18 21 17 10 24 13
14 11 25 08 14 21 13 10 23.

Translating into the ciphertext results in

ZNOYO YGYOS VRKYN
OLZIO VNKX.

To decipher, use the inverse transformation $P \equiv C - 6 \pmod{25}$, and reverse the above procedure in order to change the ciphertext back into the plaintext.

Another type of cipher is the cyclic shift cipher.[61] Here, a number called the key, is repeated in order to transform the plaintext into the ciphertext. The mathematical expression for this procedure is $C_n \equiv P_n + k_{n \pmod{M}} \pmod{26}$ where M is the number of digits in the key.

An example of a cyclic shift cipher with the plaintext, a key of 125, and the ciphertext would be

THIS	IS	A	CYCLIC	SHIFT	CIPHER.
1251	25	1	251251	25125	125125
UJNT	KX	B	EDDNND	UMJHY	DKUIGW.

A more general type of cipher is the affine transformation[62] which can be expressed mathematically by $C \equiv aP + b \pmod{26}$. It is required that $(a,26) = 1$ so that as P runs through a complete system of residues modulo 26, C also does. Note that when $a = 1$ and $b = 0$, a simple shift cipher results. The inverse relationship would be expressed by $P \equiv \bar{a}(C - b) \pmod{26}$.

As an example, let $a = 7$ and $b = 10$. Then, the enciphering equation would be $C \equiv 7P + 10 \pmod{26}$, and the deciphering equation would be $P \equiv 15(C - 10) \pmod{26}$, since 15 is the inverse of 7 modulo 26. The student should now construct a table, using these parameters, similar to the one constructed for the simple shift cipher.

Problem: If $a = 7$ and $b = 10$, encipher the plaintext given by :

PLEASE SEND MONEY.

Solution: LJMKG MGXFQ EXMW.

Problem: If $a = 7$ and $b = 10$, decipher the ciphertext given by:

FE XEN XMBMKJ NHM GMYZMN.

Solution: DO NOT REVEAL THIS SECRET.

Exponential ciphers were invented by Pohlig and Hellman in 1948.[63] Let (k, p - 1) = 1 where k is the enciphering key and p is an odd prime. Then, the enciphering equation is $C \equiv P^k \pmod p$, and the deciphering equation is $P \equiv C^j \pmod p$ where i is the inverse of k modulo (p - 1).

For example, let p = 2633 and k = 29 so that (k, p - 1) = (29, 2632) = 1. Let the plaintext be THIS IS AN EXAMPLE OF AN EXPONENTIAL CIPHER. Then, the letters are converted into their numerical equivalents in blocks yielding:

1907	0818	0818	0013	0423
0012	1511	0414	0500	1304
2315	1413	0413	1908	0019
0814	1302	0815	0704	1723.

The two digits 23 (corresponding to the letter X) were added to the end in order to fill out the message.

Next, convert the plaintext number equivalents into the ciphertext number equivalents using $C \equiv P^{29} \pmod{2633}$, which gives:

2199	1745	1745	1206	2437
2425	1729	1619	0935	0960
1072	1541	1701	1553	0735
2064	1351	1704	1841	1459.

To decipher using k = 29 and p = 2633, the inverse of 29 (mod 2632) is found using the division algorithm as in the section on congruences. Then j, the inverse of k (mod 2632), is found to be 2269. Then $P \equiv C^{2632} \pmod{2633}$.

Finally, public key cryptology, invented by Rivest, Shamir. and Adleman, involves transformations which are made public.[64] The key is (k,n) where k is the exponent and n is the product of two large primes such that $(k, \phi(n)) = 1$. The enciphering key can be made public because an unreasonably large amount of computer time would be required to find the deciphering transformation. The Euler function, $\phi(n)$, is defined as the number of positive integers not exceeding n which are relatively prime to n. Knowing that $\phi(p) = p - 1$ if p is prime, and that $\phi(mn) = \phi(n)\,\phi(m)$ if m and n are relatively prime, it follows that $P \equiv C^k \pmod{n}$ and $C \equiv P^j \pmod{n}$, where k and j are inverses modulo $\phi(n)$.

For example, if the two primes are 43 and 59 (these are much smaller than the primes that would normally be used), and if k = 13, then n = 43 x 59 = 2537, (13, 42 x 58) = 1, and j equals the inverse of 13 (mod ϕ(2537)) = 13 (mod 43 x 59) = 2436. After applying the division algorithm, it is found that j = 937.

Using this information, the student should encipher the following plaintext:

PUBLIC KEY CRYPTOLOGY.

The ciphertext becomes:

0095	1648	1410	1299
0811	2333	2132	0370
1185	1457	1084.	

Two useful references on cryptology are Elementary Number Theory by Kenneth H. Rosen and Mathematics--Its Magic and Mystery by Aaron Bakst. These references also deal with looking for patterns in order to break codes.

Mystic Arrays

There seems to be no particular point in time when magic squares were first noted. According to legend, a turtle was found with a magic square on its shell many centuries before the birth of Christ. The Lo Shu 3 x 3 magic square was known around 1000 B.C.[65]

A magic square is magic when the sum, called the magic constant, of all of the rows, columns, and diagonals is the same. A 3 x 3 magic square contains nine different integers; the simplest would contain the integers one through nine. The magic constant of this simplest of magic squares is found by summing all of the integers and then dividing by three. Using the formula for the sum of an arithmetic progression yields, (1 + 2 + . . . + 9)/3 = (9)(10)/2/3 = 15.

There is only one basic 3 x 3 magic square although eight patterns can be produced by making the appropriate rotations and reflections of the basic pattern. The method of construction of a 3 x 3 magic square, called the de la Loubère method[66], proceeds as follows. Write the digit 1 in the top center cell and continue to write consecutive integers, if possible, in a right-upward diagonal path. When the top row of cells is reached, enter the next digit in the bottom row, one column to the right. This is called the "knight move". When the right column of cells is reached, enter the next digit in the left column, one row up (the knight move). Neither of the first two maneuvers is possible if the top right cell is reached. When this occurs, write the next digit in the cell directly below. This is called "dropping down". Also, if a cell is occupied, then drop down.

8	1	6
3	5	7
4	9	2

FIGURE 6

A 3 x 3 Magic Square

In the above figure, after 1 is entered in the top center row, the diagonal path is impossible. So 2 is entered using the knight move. Since 2 is in the right column, 3 is placed using the knight move. Now, the diagonal path is blocked, so 4 is positioned by dropping down. The diagonal path is opened for 5 and 6, but then 7 must be entered by dropping down, since the 6 is in the upper right corner. Finally, the 8 and 9 are placed by using the knight moves.

All odd-order magic squares can be constructed using the technique described above. The student should attempt a 5 x 5 and a 7 x 7 magic square remembering to follow an upward-right diagonal path whenever possible.

Even-order magic squares are quite a different story. The sum of the rows, columns, and diagonals is equal to (16)(17)/2/4 = 34. The general method of construction is credited to de la Hire.[67]

Starting at the top left cell with the number 1, and working left to right while counting to 16, place the numbers only in the cells through which the main diagonals pass. This will give:

1	•	•	4
•	6	7	•
•	10	11	•
13	•	•	16.

Now, go back to the top row and fill in the missing numbers starting with 16 and counting backwards. The completed square will look like:

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16.

There are exactly 880 different 4 x 4 magic squares, ignoring rotations and mirror images. The earliest recorded fourth-order square, shown below, is shown in Dürer's famous 1514 engraving known as *Melancholia*.[68]

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

FIGURE 7

A Diabolic Magic Square

This square belongs to a special class called diabolic squares of which there are 48 basic types.[69] All the rows, columns, and diagonals add up to 34, just as in any ordinary 4 x 4 magic square. However, the four corner squares (16, 13, 4, 1) and the four center squares

(10, 11, 6, 7) also add up to 34; so do the opposite pairs of squares (3, 2, 15, 14 and 5, 9, 8, 12), as well as the slanting squares (2, 8, 15, 9 and 3, 5, 12, 14). Also, the four corner blocks (16, 3, 5, 10; 2, 13, 11, 8; 9, 6, 4, 15; and 7, 12, 14, 1) each sum to 34.

The sum of the numbers in the first two rows equal the sum of the numbers in the last two rows. In addition to this, the sum of the squares of these rows is also equal. Thus

$$16 + 3 + 2 + 13 + 5 + 10 + 11 + 8 =$$
$$9 + 6 + 7 + 12 + 4 + 15 + 14 + 1 = 68$$

and

$$16^2 + 3^2 + 2^2 + 13^2 + 5^2 + 10^2 + 11^2 + 8^2 =$$
$$9^2 + 6^2 + 7^2 + 12^2 + 4^2 + 15^2 + 14^2 + 1^2 = 748.$$

Additionally, the sum of the numbers of the alternate rows (first and third, second and fourth) and the sum of the squares of these numbers also add up to 68 and 748, respectively.

The same patterns can be shown for the columns. (The student should check this.)

Furthermore, the sum of the numbers in the diagonals equals the sum of the numbers not in the diagonals; the same can be said for the sums of the squares and cubes of these numbers. Thus:

$$16 + 10 + 7 + 1 + 4 + 6 + 11 + 13 =$$
$$5 + 3 + 2 + 8 + 12 + 14 + 15 + 9 = 68, \text{ and}$$

$$16^2 + 10^2 + 7^2 + 1^2 + 4^2 + 6^2 + 11^2 + 13^2 =$$
$$5^2 + 3^2 + 2^2 + 8^2 + 12^2 + 14^2 + 15^2 + 9^2 = 748, \text{ and}$$

$$16^3 + 10^3 + 7^3 + 1^3 + 4^3 + 6^3 + 11^3 + 13^3 =$$
$$5^3 + 3^3 + 2^3 + 8^3 + 12^3 + 14^3 + 15^3 + 9^3 = 9248.$$

Note also that the sums of the numbers in the opposite slanting cells, their squares, and their cubes are equal. Thus:

$$2 + 8 + 9 + 15 = 3 + 5 + 12 + 14 = 34, \text{ and}$$

$$2^2 + 8^2 + 9^2 + 15^2 = 3^2 + 5^2 + 12^2 + 14^2 = 374, \text{ and}$$

$$2^3 + 8^3 + 9^3 + 15^3 = 3^3 + 5^3 + 12^3 + 14^3 = 4624.$$

Finally, the date of the painting, 1514, can be found in the bottom row of the square.[70]

The student is encouraged to investigate 6 x 6 and 8 x 8 magic squares. Information on the methods of forming these mystic arrays can be found in the following books: Mathematical Diversions by J. A. H. Hunter and Joseph S. Madachy, Mathematical Puzzles and Diversions by Martin Gardner, and Fun with Mathematics by Jerome S. Meyer.

Root Extraction

Extracting Square Roots

The operations in finding the square root of a number without the use of a calculator are based upon principles found in algebra.[71] The root is represented by a binomial of the form a + b with a an integer and b the remainder. The square is of the form $a^2 + 2ab + b^2$. The integer a is found by successive approximations. To illustrate, find the square root of 207,936.

```
   4  5  6
 √20 79 36
  16
   4 79              80
   4 25               5
     54 36           85          900
     54 36                         6
         0                       906
```

1. Divide the number to be squared into groups of two digits each, from right to left.
2. Find the largest perfect square that is less than or equal to the first group (20). Write that perfect square (16) below the group and its square root (4) above the group.
3. Subtract the perfect square from the first group, then bring down the next group to make the first remainder (479).
4. In a separate memorandum column to the right, double the quotient, thus far, and add one zero (80).
5. Now, estimate the digit (5) that must be added to the memo number (80) so that when the resulting sum (85) is multiplied by that digit (5), the product (425) will be the

largest possible without exceeding the first remainder (479). Write this digit (5), when correctly estimated, as the second digit of the quotient.

6. Write the product just found (425) under the first remainder, subtract, and bring down the next group of two to form the second remainder.

7. Repeat steps 4, 5, and 6 until all of the groups are exhausted. If the original number is not a perfect square, mark the decimal and add as many zeros, in groups of two, as needed for desired accuracy.

In order to see why the algorithm works, recall that the square of a number is of the form $a^2 + 2ab + b^2$, while the root is of the form $a + b$, with *a* being the integral part of the root and*b* being the remainder. Then, when 4 was written as the first digit of the root, the implication was that $a = 400$. When a^2 was subtracted from the original number, the remainder was of the form $2ab + b^2$, which is treated as $b(2a + b)$. It can now be seen why *a* was doubled and added to *b*, the sum then being multiplied by *b*.. (Incidently, the 80 written in the memo is really 800, but the zero is dropped for convenience.)

Having completed these operations, the new quotient becomes a new *a* , and then a new *b* becomes the next digit of the root, and so on.

Extracting Cube Roots

The operations in finding a cube root are based on similar algebraic principles.[72] The number is of the form $a^3 + 3a^2b + 3ab^2 + b^3$ while the cube root is of the form $a + b$. To illustrate, find the cube root of 76,765,625.

```
  4   2   5
∛76 765 625
 64                     4800
 12  765                 240          529200
 10  088                   4            6300
  2  677  625           5044              25
  2  677  625                         535525
            0
```

1. Divide the number to be cubed into groups of three digits each, from right to left.
2. Find the largest perfect cube that is less than or equal to the first group. Write that perfect cube (64) below the group and its cube root (4) above the group.
3. Subtract the perfect cube from the first group, then bring down the next group to make the first remainder (12765).
4. In a separate memorandum column to the right, triple the square of the quotient, thus far, and add two zeros (4800).
5. Now, estimate the digit (2) whose square (4) must be added to the sum of the memo number (4800) and the product of three times the quotient thus far (4) times the digit (2) with a zero added (240), so that when the resulting sum (5044) is multiplied by that digit (2), the product (10088) will be the largest possible without exceeding the first remainder (12765). Write this digit (2), when correctly estimated, as the second digit of the quotient.
6. Write the product just found (10088) under the first remainder, subtract, and bring down the next group of three to form the second remainder.
7. Repeat steps 4, 5, and 6 until all of the groups are exhausted. If the original number is not a perfect cube, mark the decimal and add as many zeros, in groups of three, as needed for desired accuracy.

In order to see why this algorithm works, the student should recall that the cube of a number is of the form $a^3 + 3a^2b + 3ab^2 + b^3$, while the root is of the form a + b, with *a* being the integral part of the root and *b* being the remainder. Then, when 4 was written as the first digit of the root, the implication was that a = 400. When a^3 was subtracted from the original number, the remainder was of the form $3a^2b + 3ab^2 + b^3$, which is treated as $b(3a^2 + 3ab + b^2)$. It can now be seen why *a* was squared, and added to 3ab and b^2, the sum then being multiplied by *b*. (Incidently, the 4800 written in the memo is really the number 4,800,000, but the three zeros are dropped for convenience.)

Having completed these operations, the new quotient becomes a new a , and then a new b becomes the next digit of the root, and so on.

It is recommended that the student, using the principles of algebra, develop an algorithm for finding fourth roots without a calculator.

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CHAPTER 5

Summary, Conclusions, and Recommendations

Introduction

A course in mathematics appreciation has been developed using historical research because a need was perceived for this type of mathematics elective at the senior high school level. High school students are not typically introduced to many of the topics chosen for this course. In addition, high school students are rarely exposed to mathematical topics through a combination of historic, recreational, and practical lenses. Both the topics chosen and the multifaceted approach in this study are supported by research.

Also, high school students are not normally afforded the opportunity to study mathematics for its own sake. The topics chosen were selected with the hope that they would lend themselves toward that end. The philosophy which pervades throughout this study is supported by NCTM in their Commission on Standards for School Mathematics.

Summary of Findings

The topics chosen for the mathematics appreciation course were found in virtually every mathematics book on number theory or recreational problems. In the section on numeration systems, students are introduced to: converting between various bases; adding, subtracting, multiplying, and dividing in various numeration systems; and two recreational anecdotes. Various applications to congruence are studied including linear congruence, the Chinese Remainder Theorem, remainders of large numbers, ISBN numbers, designing round robin tournaments, digital roots, casting out nines, deriving the divisibility rules of selected numbers, and an anecdotal problem. Diophantine equations are divided into linear and nonlinear types including applications such as Pythagorean triples, integers as the sum of two squares, and the famous Pell equation. In the study of the Fibonacci sequence, the student is presented with the concept of recursion as well as the relationship of the Fibonacci to Pascal's triangle and to nature. The derivation of the golden section is presented, and its occurrence in geometric shapes and in the Fibonacci sequence is noted. The properties of imaginary numbers and their application to finding all of the nth roots of a number are developed. The concept of the exponential function and its representation as a power series is examined. Students are encouraged to attempt the famous Buffon experiment dealing with the experimental determination of the value of π. Perfect numbers and their relationship with Mersenne primes are studied, along with some curious oddities. The fact that numbers have shape can be seen by the student as a result of having studied triangular numbers and their many families. Various types of ciphers are examined, and the student is encouraged to perform further research on enciphering and deciphering. Algorithms for both odd and even magic squares are shown, and it is recommended that the student construct some higher order mystic arrays. Finally, the algorithms for square root extraction and cube root extraction are explained, and the relationship between the extraction techniques and the binomial expansion is emphasized.

Conclusions

The frequency of occurrence of the topics chosen in this study appears to indicate their value. It is the conclusion of the author that these topics will serve to open up a whole new world of mathematics for the interested high school student. The topics are presented utilizing an approach to mathematics to which the typical high school student might not otherwise be exposed. It is therefore hoped that the mathematics appreciation course will serve to enhance the repertoire of the serious high school mathematics student.

Recommendations

Each of the topics covered in this study can be expanded, depending upon the interest and expertise of the class. In addition, a mathematics appreciation course such as the one presented here could serve as a second semester complement to a first semester course on number theory at the college level. Also, many of the topics, such as Fibonacci sequences and numeration systems could prove quite useful to computer science students because of their applications to computer programming. Finally, because there is approximately one month available to students after having taken their advanced placement exams, the author suggests that advanced placement physics teachers could successfully utilize many of the topics presented in this course during this time, as well as throughout the school year as an alternative physics laboratory.

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